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PLANE TRIGONOMETRY

PART I

WITH THE USE OF LOGARITHMS

BY THE REV. J. W. COLENSO, M.A.
RECTOR OF FORNCETT ST. MARY, NORFOLK,
LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

LONGMAN AND CO., LONDON.

MDCCCLI.

183. g. 19.

CAMBRIDGE:
PRINTED BY METCALFE AND PALMER.

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Part II *will be published shortly, and will contain the higher parts of Trigonometry, with a large collection of Miscellaneous Examples, in all parts of the Subject, arranged progressively in order of difficulty.*

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ERRATA.

Page 12, line 13 from bottom, for *regular* read *angular*.
" 42, line 4 and 5 from bottom, dele *coversed-sine and*.
" 104, Art. 127, (3): the figure has been accidentally omitted; but the Student can easily draw it, the perpendiculars *BE*, *CF*, being supposed, in the first instance, to fall *both within* the extremities of the side *AD*.

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PLANE TRIGONOMETRY.

CHAPTER I.

ON THE MEASUREMENT OF LINES AND ANGLES.

1. **Trigonometry**, from *τρίγωνον*, *a triangle*, and *μετρέω*, *I measure*, means properly the science which treats of the *measurement of triangles*, that is, of their sides, angles, areas, &c.; being called either *plane* or *spherical* trigonometry, according as the triangles are drawn upon a plane or on the surface of a sphere, in which latter case the sides will be circular arcs and the angles curvilineal.

But the word is now used in a much more extended sense, so as to include all manner of *algebraical* reasoning about lines and angles, whether parts of a triangle or not, and, particularly, when that reasoning is carried on by means of certain quantities, which are called the trigonometrical *Ratios* or *Functions* of an angle.

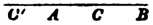
Of these Ratios we shall speak presently, so far as the subject of plane trigonometry is concerned. But we must first make some remarks upon the mode in which lines and angles are represented for the purposes of algebraical reasoning.

2. A line is represented algebraically by some letter, as a, which denotes the number of times it contains a certain fixed line, taken as the *unit* of measurement.

Thus if the unit be a *foot* or an *inch*, then the line a would be one *containing* a *feet* or a *inches*.

3. Here, however, it will be seen that we have unconsciously introduced the idea of a line having a *positive* or *negative* value. For a, we know, means $+a$; and, if in our reasonings we should happen to arrive at some negative result as the value of a line—if, for instance, it came out, at the end of a problem, that the value of a certain line was $-b$—the question arises, 'What would this (–) sign mean?'

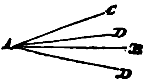
We shall shew that it indicates *contrariety of direction* to some line which we have before represented by a positive algebraical symbol.



For let AB be a line containing a units, and which we, therefore, express algebraically by a or $+a$. Then, if from the end B we measure off in the direction BA a line BC containing b units, the length of the line AC will be $a-b$ units, and, therefore, its algebraical expression will be $a-b$. Now, if b be less than a, this expression for AC will be *positive*, and C will also lie to the right of A, that is, AC will be measured in the *same* direction as AB. But, if b be greater than a, then C will fall to the left of A as at C', and AC' will be measured in the *contrary* direction to AB, its value in units being $b-a$; while, at the same time, the expression $a-b$ will become *negative*, and may be written $-(b-a)$.

We infer then that, when the algebraical expression for a line is negative, the (–) sign indicates contrariety of direction to one which has been already assumed to be positive; and, conversely, that if a line drawn in any one direction from a given point be considered positive, when represented algebraically, then a line drawn from the same point in the opposite direction must be considered negative.

4. An *angle* in Geometry is defined to be 'the inclination of two straight lines to one another,' and must, therefore, be always of necessity less than two right angles. And the same will be true in trigonometry, properly so called, where we shall have only to deal with angles, which are actually angles of triangles. But, in the wider sense of the word, an angle may be imagined to be of any magnitude whatever, being conceived to be described by the revolution of a straight line about a given point from one position to another.



Thus the angle BAC may be conceived to have been described by the revolution of AC about A from the position AB to its present position: but this it may have done by going one or more times completely round, describing in each complete revolution *four right angles* about A.

5. Here also we may shew, just exactly as in (3), that if an angle, formed upon one side of a line, by a line revolving in one direction, be considered *positive*, when expressed algebraically, then an angle, formed upon the other side of it, by a line revolving in the opposite direction, must be considered to be *negative*.

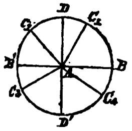
For let BAC be an angle, formed upon one side of AB, which we denote algebraically by A or $+A$; and from AC take off in the opposite direction an angle $CAD = B$. Then the algebraical expression for the angle BAD will be $A - B$, and this will be positive or negative, according as A is greater or less than B, that is, according as BAD is formed on the same side of AB with the positive angle BAC, or on the other side of AB, as BAD'; in which latter case, its algebraical expression may be written $-(B - A)$, where $B - A$ represents the actual *magnitude* of the angle, and the $(-)$ sign indicates *its position with* regard to the positive angle BAC.

6. It will be seen, as we proceed, more fully, that the above principle, by which we are led to express by algebraical symbols the *direction* as well as the *magnitude* of lines and angles in our figures, is one of the utmost importance in trigonometry, as in all Algebraic Geometry. Its usefulness, however, will appear by what follows.

In solving a problem by common Geometry, it often happens that a difference in the figure *may* lead to a difference in the result. If, for instance, we have drawn a certain angle *acute*, when we might have drawn it *obtuse*—or upon one side of a line, when we might have taken it upon the other—we cannot always be sure that the result we have obtained with the one angle would also hold good with the other. Hence we are often obliged in pure Geometry to examine different cases of the same problem, and sometimes to state different modifications of the same result. An instance of this occurs in Euc. II. 12, 13, where the square of the side subtending an angle of a triangle is *greater* or *less* than the sum of the squares of the sides containing it, according as the angle is obtuse or acute; and two different propositions are required to prove this, the latter also being divided into three cases.

Now in Algebraic Geometry it is found that we need not trouble ourselves with any consideration of these different cases: for any general figure that we draw, consistent with the conditions of the problem, will lead us to a result, that shall be true for all possible cases comprehended in it, if we only pay attention to the above Rules with respect to the signs of lines and angles, *of the same kind but drawn in contrary directions*, and interpret our results also in accordance with our assumptions. We are led to these Rules by such reasoning as that in (3) and (5): but the truth of the statement here made may be considered to be most satisfactorily proved by the perfect agreement of innumerable results, obtained by means of it, of all possible kinds and degrees of complexity, with those which might be otherwise arrived at by the more laborious processes of common Geometry.

7. In the annexed figure, BB', DD', are drawn, intersecting at right angles in A; and AC_1, AC_2, AC_3, AC_4, are supposed to be different positions of the line AC, revolving about A from a position coincident with AB, in the direction from B to D, which it is usual to consider the *positive* direction. It is plain that the extremities of these lines will all lie in the circumference of a circle, whose centre is A.



By the lines BB', DD', the whole angular space about A is divided into four equal parts, BAD, DAB', $B'AD'$, $D'AB$, which are called respectively the *first*, *second*, *third*, and *fourth quadrants*.

When the line begins to revolve from AB, it may be supposed to have made with AB an angle *zero;* when it reaches the position AC_1, it will have described an angle BAC_1 less than a right angle, which is said to be an angle *in the first quadrant;* when it comes to coincide with AD, it will have described the right angle BAD; when it reaches the position AC_2, it will have described the angle BAC_2, greater than one right angle and less than two, which is said to be an angle *in the second quadrant;* and, when it comes to coincide with AB', it will have described the *trigonometrical* angle BAB', *or two right* angles.

In like manner, the angle BAC_3, greater than two and less than three right angles, is an angle *in the third quadrant;* the angle BAD' is an angle of three right angles; the angle BAC_4, greater than three and less than four right angles, is an angle *in the fourth quadrant;* and, when the line becomes again coincident with AB, it will have described a trigonometrical angle of four right angles.

And now it may still be supposed to go on revolving, and so come again to the position AC, in the first quadrant, having described an angle greater than four and less than five right angles: and so on.

So also, the above being all *positive* angles, the line may be supposed to revolve in the opposite direction, and form successively the *negative* angles BAC_4, BAC_3, &c. It will be found, however, that all the trigonometrical properties of the angle BAC_4 will be the same, whether we suppose it to be the *positive* angle, greater than three right angles, or the *negative* angle, less than one, except, of course, such properties as may depend upon actual *magnitude,* and not merely upon the amount of angular opening between the lines containing it.

8. In Euc. I. Def. 10, we are told that right angles are formed, 'whenever one straight line, falling upon another, makes the adjacent angles *equal.*' This defines completely the magnitude of a right angle, which, being thus known, may now be taken as a measure of other angular magnitudes: just as we take a known length, a *foot,* for instance, as a measure of other lengths, only observing that a foot is an arbitrary length, fixed merely by Act of Parliament, whereas the right angle is accurately determined by its very definition.

9. Although it is usual, in treatises on trigonometry, to take AB as the starting, or *initial*, line, when, as in this case, we are representing in one figure angles of different magnitudes, yet, of course, we might reckon our angles from any other line, or some angles from one line, and others from another. In all such cases, each angle would be in the first, second, &c. quadrant, reckoning the quadrants from its own *initial-line*, and would be positive or negative, according to the direction we have chosen to be the positive direction for angles measured from that same initial-line. Thus if we chose to measure angles from AD, and call them positive in the direction DB, then DAC_1 would be a positive angle in the first quadrant, and DAC_2 would be a positive angle in the fourth quadrant, or a negative angle in the first.

10. The right angle is divided into 90 equal parts, called *degrees*, each degree into 60 *minutes*, each minute into 60 *seconds*, angles less than a second being usually expressed as decimals of a second. The marks ° ′ ″ are used to express these divisions respectively.

Thus $59° 14' 57''.85$ denotes an angle containing 59 *degrees*, 14 *minutes*, and 57.85 *seconds*.

Hence it follows that in one right angle there are 90°, in two right angles 180°, in three right angles 270°, and in four right angles 360°, &c. So also $\frac{1}{3}$ of a right angle is 30°, $\frac{1}{2}$ of a right angle is 45°, $\frac{2}{3}$ of a right angle is 60°, &c.

11. The *complement* of an angle is its defect from a right angle, or comp $(A) = 90° - A$.

The *supplement* of an angle is its defect from two right angles, or supp $(A) = 180° - A$.

Thus comp $(23° 23' 27'') = 66° 36' 33''$, comp $(123° 13' 27'') = -33° 13' 27''$, *supp* $(45°) = 135°$, supp $(-45°) = 225°$

12. Hence in the figure of (7), if $BAC_1 = 75°$, then its complement $= 90° - 75° = 15° = DAC_1$; and if $BAC_2 = 130°$, then its complement $= 90° - 130° = -40° = DAC_2$ the angle being here formed upon the *negative* side of AD, since, in taking DAC_1 as a *positive* angle, we have made the direction from D towards B, in which DAC_1 increases, the positive direction for all complements, measured from AD. So likewise the supplement of BAC_2 is $B'AC_2$, and that of BAC_3 is $B'AC_3$ which latter is measured on that side of AB', which though positive for the angle BAC_3 itself, is negative for its supplement.

To avoid the awkwardness of having the same direction positive for one set of angles and negative for others, it will be found hereafter generally more convenient to measure the complement and supplement as well as the angle itself from AB, all in the same direction.

13. Since the three angles of any triangle are together equal to two right angles, it follows that

(1) Each acute angle of a *right-angled* triangle is the complement of the other;

(2) Each angle of *any* triangle is the supplement of the sum of the other two angles.

We see also that if A, B, C, represent the three angles of any triangle ABC, then $A + B + C = 180°$; and, besides this, we only require two other independent equations to be given between the three angles in order to determine them completely.

Ex. The angles of a triangle are in A. P., and the difference between the greatest and least is 20°: find them.

Here (1) $A + B + C = 180°$, (2) $A - B = B - C$, (3) $A - C = 20°$
from (2) $A + C = 2B$, and $\therefore$ from (1) $3B = 180°$, or $B = 60°$
hence $A + C = 120°$, and (3) $A - C = 20°$;
$\therefore 2A = 140°$, $2C = 100°$, or $A = 70°$, $C = 50°$.

Ex. 1.

1. Write down the complements of $15^\circ\,37'\,2''$, $142^\circ\,15'\,29''$, $1'\,3''.2$.

2. Write down the supplements of $145^\circ\,15'\,20''$ and $-27^\circ\,17'\,7''$.

3. If one angle of a right-angled triangle is $9^\circ\,9'\,9''$, what will e other be? Express them both in *degrees* only.

4. Given the difference of the two acute angles of a right-gled triangle to be $24^\circ\,24'$, find the two angles.

5. The base angle of an isosceles triangle is $59^\circ\,59'\,59''$: find e vertical angle.

6. The vertical angle of an isosceles triangle is a third as large ain as either of the base angles: find them.

7. The semi-sum of two angles of a triangle is 60°, and their mi-difference is 10°: find the three angles.

8. The exterior angle of a triangle is half as large again as one the two interior and opposite angles, and its supplement is half large again as the other: find the angles of the triangle.

9. The angles of a right-angled triangle are in A. P.: find them.

10. The supplement of one angle of a triangle is double of the mplement of another, and triple that of the third: find all three.

14. By Laplace, however, and some other French iters, the right angle was divided into 100 equal rts called *grades*, each grade into 100 *minutes*, each inute into 100 *seconds*, which were denoted by the rks ^{g} $^{\backprime}$ $^{\backprime\backprime}$: thus $59^{g}\,14^{\backprime}\,57^{\backprime\backprime}.85$.

Of course it will be understood that the Foreign nute and second, though called by the same name, e not of the same magnitude as the English.

The advantage of this *decimal* method of dividing the ht angle, (which was introduced in France, with ner decimal divisions for *money*, *weight*, *length*, &c. the time of the French Revolution,) was that any gle, given in grades, minutes, and seconds, could mediately be written down as a decimal of a grade.

Thus the above angle $59^g\ 14'\ 57''.85$ may be written down once as $59^g.145785$: for $1' = \frac{1}{100}g$, and, therefore, $14' = \frac{14}{100}g = .14$ and $1'' = \frac{1}{10000}g$, and, therefore, $57''.85 = \frac{57.85}{10000}g = .005785g$.

And so, of course, conversely, an angle, given i grades and decimals of a grade, could be written dow at once in grades, minutes, and seconds.

Thus $43^g.012345 = 43^g\ 1'\ 23''.45$.

The use of the grade, however, is now abandone even in France, as it involved the sacrifice of suc voluminous tables and records, already adapted to th *sexagesimal* division, or else of such a vast amount time and labour, in reducing them to the decimal systen

It may be observed that the *sexagesimal* metho allows of the whole angular space about a point bein divided into a greater number of *aliquot parts* than th decimal, the divisors of 360 being 24 in number, an of 400 only 15. Thus we can express exactly in *d* *grees* $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$, $\frac{1}{12}$, &c. of a right angle, but not in *grade*

15. *To convert the English measure of an angle in the Foreign, and vice versâ.*

Questions of this kind are now merely matters of curiosity; b they may afford the Student some useful practice.

Let E, F, denote the number of *degrees* and of *grad* respectively in the *same* angle A; and let e, f, denote t angular magnitudes of a degree and grade respective

Then we have Ee = angle $A = Ff$; but we have al $90e$ = a right angle $= 100f$; $\therefore\ \frac{E}{90} = \frac{F}{100}$, or $\frac{E}{9} = $

$$\text{hence}\quad (1)\ E = \frac{9}{10}F = \left(1 - \frac{1}{10}\right)F = F - \frac{F}{10},$$

$$(2)\ F = \frac{10}{9}E = \left(1 + \frac{1}{9}\right)E = E + \frac{E}{9}.$$

From the above results we obtain the following Rules:

(1) From the number of *grades* in any angle subtract $\frac{1}{10}$th, and the result will be the number of *degrees* in the same angle;

(2) To the number of *degrees* in any angle add $\frac{1}{9}$th, and the result will be the number of *grades* in the same angle.

It should be observed that E in the above is the *number of degrees* in the given angle. If, therefore, it be originally given in degrees, minutes, and seconds, the minutes and seconds must be expressed as a decimal of a degree, before the Rule can be applied.

Ex. 1. Convert $59^g\,14^{\backprime}\,57^{\backprime\backprime}.85$ into English measure.

$$\begin{array}{lrl}
\text{Here } 59^g\,14^{\backprime}\,57^{\backprime\backprime}.85 = & 59.1457850 & \textit{grades}\\
\text{subtract } \tfrac{1}{10}\text{th} = & \underline{5.9145785} & \\
& 53.2312065 & \textit{degrees} = 53^\circ\,13'\,52''.34.\\
& \underline{60} & \qquad\quad\;\, = 53^\circ\,13'\,52''.\\
& 13.8723900 & \\
& \underline{60} & \\
& 52.34340 &
\end{array}$$

Ex. 2. Convert $59^\circ\,14'\,57''.85$ into Foreign measure.

Here $59^\circ\,14'\,57''.85 = 59.2494027$ *degrees*, as appears from below;

$$\begin{array}{lrl}
\text{add } \tfrac{1}{9}\text{th} = & \underline{6.5832669} & \\
& 65.8326696 & \textit{grades} = 65^g\,83^{\backprime}\,26^{\backprime\backprime}.69.\\
60\,\big)\;\underline{57.85000} & & \qquad\quad\;\, = 65^g\,83^{\backprime}\,27^{\backprime\backprime}.\\
60\,\big)\;\underline{14.964166} & & \\
\quad .2494027 & &
\end{array}$$

It will be seen that in Ex. 2 we add one to the number of seconds, because the neglected part .69 is greater than $\frac{1}{2}$ or .5: whereas in Ex. 1 the neglected part .34 is less than .5, and we take no notice of it. This is the usual practice in such cases, where very great accuracy is not required.

16. It appears, from Euc. I. 32, Cor. 1, that 'all the interior angles of any rectilineal figure, together

with four right angles, are equal to twice as many right angles as the figure has sides.'

Hence if n be the number of sides of any rectilineal figure, we have the sum of its n angles $+ 4 \times 90^\circ = 2n \times 90^\circ$ or the sum of its n angles $= (2n - 4)\, 90^\circ = (n - 2)\, 180^\circ$.

If the figure be also a *regular* figure, that is, equilateral and equiangular, then its n angles are all equal and, consequently, each of them $= \dfrac{n-2}{n}\, 180^\circ$.

Thus, if $n = 3$, each angle of an equilateral triangle is $\frac{1}{3} \times 180^\circ = 60^\circ$; if $n = 4$, each angle of a square is $\frac{2}{4} \times 180^\circ = 90^\circ$; &c.

17. Since (Euc. I. 15, Cor. 2) the sum of all the angles 'subtended at its centre, or *at the centre of its circumscribing circle*, by the n sides of a regular polygon, is *four right angles* or 360°, it follows that the angle subtended by each side at the *centre* is $\dfrac{360^\circ}{n}$, and at the *circumference*, $\dfrac{180^\circ}{n}$.

Ex. The angle between the two lines which join the extremities of a side of an octagon with *any one* of its regular points is $\dfrac{180^\circ}{8} = 22\frac{1}{2}^\circ$.

Ex. 2.

1. Write down the complements of -15^g, $27^g.25$, $32^{\backprime}\ 1^{\backprime\backprime}$.

2. Write down the supplements of $1^g\ 2^{\backprime}\ 3^{\backprime\backprime}$, $89^g.0004$, and $225^g.1001$.

3. Convert $37^g\ 15^{\backprime}\ 7^{\backprime\backprime}$ into English measure, find the complement of the result, and reconvert it to grades.

4. Express 1° and $1''$ in Foreign measure, and 1^g and $1^{\backprime}$ in English measure.

5. Find the interior angles of a regular pentagon and hexagon.

6. Shew that the interior angles of a regular octagon and dodecagon are as $9 : 10$.

7. If an isosceles triangle be inscribed in a circle on the side of a regular inscribed heptagon, find the ratio between its vertical and base angles.

8. The angles of a pentagon are as the numbers 2, 3, 4, 5, 6: find them.

9. In the figure of Euc. IV. 10, shew that AC is the side of a regular decagon inscribed in the larger circle, and of a regular pentagon inscribed in the smaller.

10. There are two polygons such that the difference of the sums of their interior angles is four right angles, and the ratio of the same as 5 : 3: find the number of sides in each.

11. One regular figure has twice as many sides as another, and an angle of the first is a third as large again as an angle of the second: find the interior angle in each.

12. The interior angles of a rectilineal figure are in A. P., the least is 120°, and the common difference 5°: find the number of sides.

---

18. In *equal* circles (Euc. VI. 33) 'the angles at the centre are proportional to the arcs which subtend them.'

Hence in the figure of (7) the *arcs* BC_1, BD, &c. of the *same* circle are proportional to the *angles* BAC_1, BAD, &c.: and, if the whole circumference be divided into 360 equal parts, then BD will contain 90 of these, BDB' 180, &c.; and, generally, any arc will contain just as many of these parts, as the corresponding angle contains degrees.

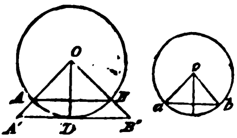
These parts of the circumference are also called *degrees*, and subdivided, as the angular degrees, into *minutes* and *seconds*, the same marks ° ′ ″ being used, as before, to denote them. And, since the *number* of degrees, minutes, and seconds in any arc will be the same as in the angle it subtends at the centre, (whatever be the radius, and, consequently, whatever may be *the actual length of* an arcual degree,) on this accoun the arc is often used as the representative, or, as it is called, the *measure* of the angle.

N.B. There is some ambiguity in the use of this word *measure*. Properly speaking, we mean by it the fixed quantity or unit, by which we measure other quantities. Then we use it also to express the number of such units contained in the measured quantity. Thus we speak both of a yard measure, and of the measure of a distance being so many yards. It is here used in a sense different from either of these, namely, of two *proportional* quantities, one of which is commonly said to measure, or be a measure of, the other.

So also BC_4, &c., measured in the opposite direction to BC_1, will be *negative* arcs; and DC_1 will be the *complement*, and $B'C_1$ the *supplement*, of BC_1.

19. *The circumferences of different circles are proportional to their radii.*

For let C, c, be the circumferences of two circles, whose radii are R, r; and suppose AB, ab, to be sides of two regular polygons, each having (n) sides, inscribed in the two circles. Then, (17), AB, ab, subtend equal angles at the centres O, o, and the triangles OAB, oab, being also isosceles, are, consequently, equiangular and similar; hence (Euc. VI. 4) $AB : ab :: AO : ao$, and, if P, p, be the perimeters of the two polygons, we have also



$$P : p :: n.AB : n.ab :: AO : ao :: R : r.$$

Now let the number of sides be increased indefinitely, their magnitudes being diminished; then the polygons tend more and more to equality with the two circles, and their perimeters with the two circumferences. Let $P = C - X$, $p = c - x$, where the differences X, x may be made as small as we please by increasing the number of [illegible] then $C - X : c - x :: R : r$, or $rC - rX = Rc - Rx$, and, therefore, $rC - Rc = rX - Rx$, where each of rX, Rx, and, therefore, their difference $rX - Rx$, may be made as small as we please, and less than any conceivable magnitude, by increasing the number of sides.

Hence $rC - Rc = 0$; for if $rC - Rc =$ any finite value a, then $rX - Rx$ could not be made less than a, which is not the case: and, therefore, $rC = Rc$, or $C : c :: R : r$, that is, the *circumference* of any circle $\propto$ its *radius*.

20. *In different circles, the arcs subtending equal angles are proportional to the radii.*

For, in different circles, the length of *one* degree, being the $\frac{1}{360}$th part of the circumference, will (19) vary as the radius; and, therefore, the arcs, which in different circles subtend at their centres the *same* angle A, and which contain, consequently, each the same number of degrees, will also be proportional to the respective radii. So that, to the same angle, the subtending arc $\propto$ radius of the circle.

21. Since the circumference of a circle $\propto$ radius, and, therefore, $\propto$ diameter, it follows (Alg. 167) that, for all circles, the ratio $\dfrac{\text{circumference}}{\text{diameter}}$ has a certain fixed numerical value. This value will be shewn hereafter to be (as far as five places of decimals) $3.14159 = 3\frac{1}{7}$ nearly, or, still more nearly, $\frac{355}{113}$; so that the circumference of any circle is rather more than three times its diameter. This number, 3.14159, is, consequently, a very important one to be remembered, as well as the fractions $3\frac{1}{7}$, $\frac{355}{113}$, which approximate to its value. (The latter of these, it may be observed, is formed of the figures 113355 in order, beginning with the denominator.) The *number* itself, 3.14159 &c., it is usual

to denote by π; but, for most common purposes, we may take for π the approximate value $3\frac{1}{7} = \frac{22}{7}$.

22. Hence, if C represent the circumference of any circle and r its radius, we shall have $\frac{C}{2r} = \pi$, or $C = 2\pi r$ = the length of 360°; and, of course, the length of 180°, or the semicircumference, is πr, that of the quadrantal arc, 90°, is $\frac{\pi}{2} r$, that of the octant, 45°, is $\frac{\pi}{4} r$, &c. The radius, however, is often not expressed; and then the expressions for the lengths of 360°, 180°, 90°, &c., are written 2π, π, $\frac{\pi}{2}$, &c.; only in such a case it must be remembered that 2π really means 2π times *the radius*, whatever that may be, and so of the rest. Since the radius is here referred to as the *unit of measurement*, the arc in such cases is said to be expressed, or measured, to *radius unity*.

23. Generally, if the length of an arc of $A°$ be θr, (as that of 180° is πr,) then $\theta r : \pi r :: A : 180$, or $\frac{\theta}{\pi} = \frac{A}{180}$, where θ, being the number of times the arc contains the radius, will be also the measure of the arc to radius unity. Hence, if A be given, then the measure of the arc to radius unity is known from the equation $\theta = \frac{A}{180}\pi$; and, conversely, if θ be given, then the number of degrees in the arc is known from the equation $A = \frac{\theta}{\pi} 180 = \theta\omega$, if we denote by ω the number $\frac{180}{\pi} = 57.29577$&c.

If the length (a) of an arc be given (in *feet*, *yards*, &c.) of course θ may be found by simply expressing the arc as a fraction of the radius; since $a = \theta r$, and $\theta = \frac{a}{r} = \frac{arc}{rad}$.

Ex. 1. Express to radius 1 an arc of 27°. *Ans.* $\theta = \frac{27}{180}\pi = \frac{3}{20}\pi$, which answer, in terms of the *known* number π, will be generally sufficient in such a case, without putting for π its numerical value.

Ex. 2. Find the measure of an arc of 10 feet to a radius of 4 feet. *Ans.* $\theta = \frac{10}{4} = 2\frac{1}{2}$, meaning, of course, that the length of the arc is (not $2\frac{1}{2}$ *feet*, but) $2\frac{1}{2}$ times the *radius*.

Ex. 3. Find the length of an arc of 10° to a radius of 10 feet.

Ans. $\theta r = \frac{10}{180}\pi r = \frac{5}{9}\pi\, ft = \frac{5}{9} \times \frac{22}{7}\, ft = 1.74\, ft.$

Ex. 4. Find the radius, when an arc of 1° measures an inch.

Here $\pi r = 180° = 180\ in = 15\, ft$; $\therefore r = \frac{15}{\pi}\, ft = \frac{105}{22}\, ft = 4.77\, ft.$

24. It is proved, in Euc. XII. 2, that 'the areas of circles are as the squares of their diameters' or radii; and the same may be thus demonstrated, as in (19).

For, referring to the figure in (19), let AB, ab, be sides of regular polygons, each having n sides, inscribed in two circles, whose centres are O, o, and radii R, r; then the triangles OAB, oab, being similar, we have (Euc. VI. 19) area OAB : area oab :: $AO^2 : ao^2$; and if A, a, be the areas of the polygons, we have also

$$A : a :: n.OAB : n.oab :: AO^2 : ao^2 :: R^2 : r^2.$$

But, when the number of sides is increased and their magnitude diminished indefinitely, the areas of these polygons approximate more and more to those of the circles; and hence, reasoning exactly as in (19),

$$\text{area of circle } ABC : \text{area of circle } abc :: R^2 : r^2.$$

25. It will now further appear that, in point of fact, the area of a circle, whose radius is R, is πR^2.

For let $A'B'$, in the same figure, be the side of a regular polygon, *circumscribing* the circle whose radius is R, and subtending the same angle at the centre as AB. Draw the radius OD to the point where $A'B'$ *touches the circle*, and, therefore, perpendicular to $A'B'$

Then, since the area of any triangle (Euc. I. 41) = $\frac{1}{2}$ area of *rectangle* of the same base and height = $\frac{1}{2}$ base × height, we have area of $A'B' = \frac{1}{2}A'B' \times OD = \frac{1}{2}A'B' \times R$; and, if P be the perimeter of the polygon, we have also

$$\text{area of polygon} = \tfrac{1}{2}(A'B' + \&\text{c.})\,R = \tfrac{1}{2}P.R.$$

But, when the number of sides is increased indefinitely, or the polygon becomes a circle, $P = 2\pi R$; and, therefore, area of circle $= \frac{1}{2}(2\pi R)\,R = \pi R^2$.

Ex. The area of a circle, whose radius is 1 foot, will be π, that is, π *square* feet, or $3\frac{1}{7}$ *sq. ft.*

The circumference will be 2π, that is, 2π *linear* feet or $6\frac{2}{7}$ *ft.*

26. Since (Euc. VI. 33) 'sectors of circles are proportional to the arcs on which they stand,' or to the angles which subtend them, it follows that the area of a sector whose arc, or angle, is 30°, 45°, 60°, &c., will be $\frac{1}{12}\pi r^2$, $\frac{1}{8}\pi r^2$, $\frac{1}{6}\pi r^2$, &c. respectively, (since the whole circumference contains 360°); and, generally, the area of a sector, whose angle is $A°$, will be $\dfrac{A}{360}\pi r^2$, or, if the circular measure of A be θ, it will be $\dfrac{\theta}{2\pi}\pi r^2 = \frac{1}{2}\theta r^2$.

Ex. 3.

1. Find the length of a degree of the meridian upon a globe of 18 inches diameter.

2. Obtain the number of degrees in a circular arc, 30 feet long, whose radius is 25 feet; and measure it to radius unity.

3. Find the measures of two arcs, of 40 feet and 40 degrees, in terms of a radius of $2\frac{1}{2}$ yards.

4. Find the thickness of a tree, which a man can exactly clasp round with three grasps of his arms, supposing him to stretch with his arms 6 feet each time.

5. Find the area of a section of the tree in the preceding question; and the quantity of wood in a length of 30 feet, supposing it of uniform thickness.

6. The inscribed square is cut out of a circle whose diameter *is a yard: find the* area of the remainder.

7. What length of cord will it take to tether an ass, so that he may graze over an acre of ground?

8. Compare the areas of two circles, when an arc of 60° in the one is equal in length to an arc of 45° in the other.

9. The hour-hand of a watch is $\frac{3}{6}$ inch long, the minute-hand $\frac{3}{4}$ inch, and the second-hand $\frac{3}{10}$ inch; compare the rates at which their extremities move.

10. The Earth is distant from the Sun 96 millions of miles, and describes her orbit round him (nearly) in 365 days; at what rate per minute does she travel?

11. The Earth being supposed a sphere, with a radius of 4000 miles, find the length of an arc of 1°, and the distance round the world on the Equator.

12. At what rate per minute does a point on the Equator move, by reason of the Earth's rotation on her axis?

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27. Let a, a', be arcs, which in different circles subtend at their centres the *same* angle A, to radii r, r', respectively; then (20) $a : a' :: r : r'$, or $a : r :: a' : r'$.

Hence, if the angle be taken such that, for any one circle, $a = r$, we shall have also for any other circle, $a' = r'$, to the *same* angle. It appears then that the angle, which is *subtended by an arc whose length is equal to the radius*, is the same for all circles whatever.

This angle, which is thus easily defined, may, like the right angle, be used as a *unit* to measure other angles, and will be found to be an angle of great importance, especially in the higher parts of the subject.

28. We may thus obtain the magnitude of the above angle in degrees, as well as the measure of any other angle when expressed in terms of this unit.

Since π denotes the number of times the circumference of any circle contains its diameter, or the *semicircumference* its *radius*, therefore, if an arc be taken equal *in length to the* radius, π is the number of times the semicircumference contains this arc, and, therefore, by (18), is also the number of times an angle of 180° contains the angle subtended by this arc. Hence the number of degrees in this angle is $\frac{180}{\pi}$, which number we may denote as before by ω; and any other angle A contains ω° just as often as its subtending arc contains the arc subtending ω°, or the radius.

Thus 180° of *arc* $= \pi r$, and 180° of *angle* $= \pi\omega^\circ$: and, if the arc $= \theta r$, then the angle $= \theta\omega^\circ = \frac{\text{arc}}{\text{rad}} \times \frac{180^\circ}{\pi}$; or each $= \theta$ simply, if we suppress, for the arc, the unit r, and, for the angle, the unit ω°.

29. The angle $\frac{180^\circ}{\pi} = 180^\circ \times \frac{113}{355} = \frac{4068^\circ}{71} = 57^\circ.29577$ $(= 57^\circ\ 17'\ 45''$ or $206265'')$ is commonly called the *unit of circular measurement*, or the *circular unit;* and the fraction $\frac{\text{arc}}{\text{rad}}$ is called the *circular measure* of the angle. It is usual to denote the circular measure of an angle, or the measure of its arc to radius unity, by some Greek letter, as θ: so that, as before in (23), $\theta = \frac{a}{r}$, or $a = r\theta$.

Thus, if for any angle A, the length of the subtending arc in any circle be three times that of the radius, then for this angle we have $\theta = 3$, and A contains 3 circular units $= 3 \times 57^\circ.29577 = 172^\circ$.

30. Hence, since $180^\circ = \pi\omega^\circ$, the circular measure of 180° is π, and those of 360°, 90°, &c. are 2π, $\frac{\pi}{2}$, &c., the same, of course, as the measures (22) of the corresponding arcs to radius unity. And, generally, as in (23), if θ be the circular measure of an angle of A°, (*as* π *is* the circular measure of an angle of 180°,) then

$\theta : \pi :: A : 180$, or $\frac{\theta}{\pi} = \frac{A}{180}$; hence, when A, the number of degrees is given, we have $\theta = \frac{A}{180}\pi$, or, when θ, the circular measure is given, we have $A = \frac{\theta}{\pi}180$, which may be written most conveniently $A = \theta\frac{180}{\pi}$, because we know the value of this fraction, viz. 57.29577, and shall only have, in any case, to multiply this by θ.

Since 200 *grades* = 180 *degrees*, it follows that the number of grades in an arc equal to the radius, or in its subtending angle, is $\frac{200}{\pi}$; and $A = \theta\frac{200}{\pi}$, $\theta = \frac{A}{200}\pi$.

Ex. 1. Find the circular measure of 35°. *Ans.* $\theta = \frac{35}{180}\pi = \frac{7}{36}\pi$.

Ex. 2. Express the angle $\frac{1}{1000}$ in degrees, &c.

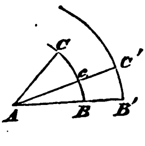
Ans. $A = \frac{206265''}{1000} = 206''.265 = 3'\ 26''.3$.

31. *In different circles, angles are directly proportional to the arcs which subtend them and inversely to the radii.*

This follows at once as a Corollary from (29); since, if θ, θ', be the circular measures of two angles A, A', subtended by arcs a, a', to radii r, r', then $\theta = \frac{a}{r}$, $\theta' = \frac{a'}{r'}$, and $A : A' :: \theta : \theta' :: \frac{a}{r} : \frac{a'}{r'}$, or angle $\propto \frac{\text{arc}}{\text{rad}}$.

But it may be well to give the following direct Proof.

Let BC, $B'C'$, be arcs subtending angles BAC, $B'AC'$, to radii AB, AB', respectively, the circles being supposed concentric without affecting the demonstration; and let the larger radius AC' cut the inner circumference in c; then (Euc. vi. 33)



$$\angle BAC : \angle BAc \text{ (or } \angle B'AC') :: \text{arc } BC : \text{arc } Bc$$
$$:: \frac{\text{arc } BC}{\text{rad } AB} : \frac{\text{arc } Bc}{\text{rad } AB} :: \frac{\text{arc } BC}{\text{rad } AB} : \frac{\text{arc } B'C'}{\text{rad } AB'} \text{ by (20).}$$

Ex. 4.

1. Obtain the circular measures of 72°, its complement, and its supplement.

2. Express 6° 3′ 36″ in circular measure.

3. The angles of a triangle are as 3, 4, 5; express them each in circular measure.

4. Express, both in circular measure and degrees, the angle subtended by an arc of π feet to a radius of π yards.

5. Find the number of degrees, minutes, and seconds in the angle .1.

6. Express $2^g\ 18^{`}\ 75^{``}$ in circular measure.

7. Compare the magnitudes of two angles, when the arcs, which subtend them, are inversely as the radii and directly as the numbers $\frac{2}{3}$ and $\frac{3}{5}$.

8. Find the complement of $\frac{\pi^\circ}{2}$, and the circular measure of the supplement of $2\pi^\circ$.

9. The measure of an arc is $3\frac{3}{4}$ to radius $2\frac{2}{3}$; what would be the measure of an arc of the same length to radius $2\frac{1}{2}$? Express also in each case the angle at the centre in degrees.

10. The length of an arc of 45° in one circle being equal to that of 60° in another, find the circular measure of the angle, that would be subtended at the centre of the first by an arc equal to the radius of the second.

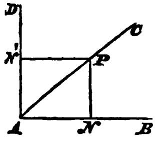
11. If a be the arc which measures the complement of an angle to radius r, find the arc which measures the supplement of the same angle to radius r'.

12. The diameter of one circle is equal in length to the semicircumference of another; how many degrees of the first are equal in length to the diameter of the second?

CHAPTER II.

ON THE TRIGONOMETRICAL RATIOS OF AN ANGLE.

32. **The** following are the quantities, which have been already spoken of under the name of the *Trigonometrical Ratios*.



Let BAC be any angle (A) in the first quadrant; from any point P in AC, one of the lines containing the angle, draw PN perpendicular upon the other line AB. Then

(1) $\frac{PN}{AP}$, or $\frac{\text{perp.}}{\text{hyp.}}$, is the *sine* of $\angle A$, or $\sin A = \frac{PN}{AP}$;

(2) $\frac{AN}{AP}$, or $\frac{\text{base}}{\text{hyp.}}$, is the *cosine* of $\angle A$, or $\cos A = \frac{AN}{AP}$;

(3) $\frac{PN}{AN}$, or $\frac{\text{perp.}}{\text{base}}$, is the *tangent* of $\angle A$, or $\tan A = \frac{PN}{AN}$;

(4) $\frac{AN}{PN}$, or $\frac{\text{base}}{\text{perp.}}$, is the *cotangent* of $\angle A$, or $\cot A = \frac{AN}{PN}$;

(5) $\frac{AP}{AN}$, or $\frac{\text{hyp.}}{\text{base}}$, is the *secant* of $\angle A$, or $\sec A = \frac{AP}{AN}$;

(6) $\frac{AP}{PN}$, or $\frac{\text{hyp.}}{\text{perp.}}$, is the *cosecant* of $\angle A$, or $\operatorname{cosec} A = \frac{AP}{PN}$.

Also the quantity $1 - \cos A$ is called the *versed-sine* of A, or $\operatorname{vers} A = 1 - \cos A$; and, of course, $\cos A = 1 - \operatorname{vers} A$.

The Student must not pass on till he has made himself thoroughly conversant with the above definitions, and able, without hesitation, to answer to such questions as 'What is the secant of A? the sine? the cotangent?

Let him notice also distinctly that the values of thes quantities are mere *numbers*, being the ratios of certai lines to one another.

Ex. If the length of PN be 3, (that is, of course, 3 units o length, whatever the unit may be,) and that of AN be 4, the $AP = \sqrt{(16+9)} = 5$; and, therefore, $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$

33. It is plain that (1) and (6), (2) and (5), (3) and (4) are the *inverse*, or *reciprocals*, of each other; so tha we shall have at once

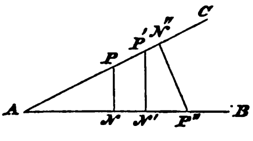
$$\sin A = \frac{1}{\operatorname{cosec} A}, \quad \cos A = \frac{1}{\sec A}, \quad \tan A = \frac{1}{\cot A}.$$

$$\cot A = \frac{1}{\tan A}, \quad \sec A = \frac{1}{\cos A}, \quad \operatorname{cosec} A = \frac{1}{\sin A}:$$

and, consequently, whenever we have found the *sine* *cosine*, and *tangent* of an angle, we may consider tha we have found all its Ratios.

34. It will be easily seen that the numerical value of the different Ratios depend only upon the magnitud of A, and not at all upon the position of the assume point P.

For if in AC, as above, we take, besides P, som other point P', and draw $P'N$ perpendicular to AB, then th triangles APN, $AP'N'$, bein similar, will (Euc. VI. 4) 'hav the sides about their equal angle proportional.' Hence the ratios $\frac{PN}{AP}$, $\frac{P'N'}{AP'}$, will b equal, or $\sin A$ will have the same value, according t our definition, wherever we take the point P in AC and similarly for the cosine, tangent, &c.



So, likewise, if we take a point P'' in AB, (instead of AC,) and draw $P''N''$ perpendicular on AC, the triangles APN, $AP''N''$, will also be similar, and, as before, the two expressions for $\sin A$, namely, $\frac{PN}{AP}$, $\frac{P''N''}{AP''}$, will still be equal: and so of the other Ratios.

It appears then that for any one angle there is but one set of trigonometrical Ratios; so that, if the angle be given, the numerical values of the corresponding Ratios are *determinate*, and, as will be shewn hereafter, can in all cases be found. Such values have been calculated for all angles from 0° to 45°, (by which, as will appear, those of all angles whatsoever can be immediately obtained,) and are registered in Tables for use.

35. As PN, the side *opposite*, is the perpendicular, and AN, the side *adjacent*, is the base, in forming the Ratios for the angle BAC or PAN, so AN is the perpendicular and PN the base, in forming the Ratios for the angle APN, which (11) is the complement of PAN or A. Hence we have

$$\sin APN = \frac{AN}{AP} = \cos PAN, \text{ or } \sin(90° - A) = \cos A,$$

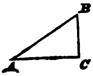
$$\cos APN = \frac{PN}{AP} = \sin PAN, \text{ or } \cos(90° - A) = \sin A.$$

And so, generally, the *sine, tangent,* and *secant* of the complement are the *cosine, cotangent,* and *cosecant* of the angle itself; and, conversely, the *cosine, cotangent,* and *cosecant* of the complement are the *sine, tangent,* and *secant* of the angle. We may express this by saying that any Ratio or Function of an angle is the corresponding *Co-ratio* or *Co-function* of its complement.

So likewise,

$$\text{covers}A = \text{vers}(90° - A) = 1 - \cos(90° - A) = 1 - \sin A.$$

36. The Student should now accustom himself to express with ease any side of a right-angled triangle in terms of any other side, by means of a trigonometrical Ratio; and it will be found that each side may be expressed, in terms of the other sides and angles, in four different ways.



Thus if ABC be a triangle, right-angled at C, we have

$$\frac{BC}{AB} = \sin A \text{ or } \cos B, \text{ and } BC = AB \sin A \text{ or } AB \cos B;$$

$$\frac{BC}{AC} = \tan A \text{ or } \cot B, \text{ and } BC = AC \tan A \text{ or } AC \cot B.$$

In like manner,

$$AC = AB \cos A = AB \sin B = BC \cot A = BC \tan B;$$

$$AB = AC \sec A = AC \operatorname{cosec} B = BC \operatorname{cosec} A = BC \sec B.$$

37. By reference to the definitions in (32), the following results may be easily obtained:

$$\tan A = \frac{PN}{AN} = \frac{PN}{AP} \div \frac{AN}{AP} = \frac{\sin A}{\cos A};$$

$$\cot A = \frac{AN}{PN} = \frac{AN}{AP} \div \frac{PN}{AP} = \frac{\cos A}{\sin A}:$$

$$\sin^2 A + \cos^2 A = \left(\frac{PN}{AP}\right)^2 + \left(\frac{AN}{AP}\right)^2 = \frac{PN^2 + AN^2}{AP^2} = 1;$$

whence $\sin^2 A = 1 - \cos^2 A$, $\cos^2 A = 1 - \sin^2 A$,

or $\sin A = \sqrt{(1 - \cos^2 A)}$, $\cos A = \sqrt{(1 - \sin^2 A)}$:

$$\sec^2 A = \left(\frac{AP}{AN}\right)^2 = \frac{AN^2 + PN^2}{AN^2} = 1 + \left(\frac{PN}{AN}\right)^2 = 1 + \tan^2 A;$$

whence $\sec A = \sqrt{(1 + \tan^2 A)}$, and $\tan A = \sqrt{(\sec^2 A - 1)}$:

$$\operatorname{cosec}^2 A = \left(\frac{AP}{PN}\right)^2 = \frac{PN^2 + AN^2}{PN^2} = 1 + \left(\frac{AN}{PN}\right)^2 = 1 + \cot^2 A;$$

whence $\operatorname{cosec} A = \sqrt{(1 + \cot^2 A)}$, and $\cot A = \sqrt{(\operatorname{cosec}^2 A - 1)}$.

38. We will here collect the preceding results, all of which must be carefully remembered, so as to be forthcoming, when required, without a moment's hesitation. For shortness' sake we give them in the following form, without mentioning the angle to which the Ratios belong, since A in the proof stands for any angle whatever.

(i) $\sin = \dfrac{1}{\text{cosec}}$, $\cos = \dfrac{1}{\sec}$, $\tan = \dfrac{1}{\cot}$, $\cot = \dfrac{1}{\tan}$,

$\sec = \dfrac{1}{\cos}$, $\text{cosec} = \dfrac{1}{\sin}$. (ii) $\tan = \dfrac{\sin}{\cos}$, $\cot = \dfrac{\cos}{\sin}$.

(iii) $\sin^2 + \cos^2 = 1$, $\sin^2 = 1 - \cos^2$, $\cos^2 = 1 - \sin^2$.

(iv) $\sec^2 = 1 + \tan^2$, $\tan^2 = \sec^2 - 1$.

(v) $\text{cosec}^2 = 1 + \cot^2$, $\cot^2 = \text{cosec}^2 - 1$.

And to these, of course, must be added

$\sin A = \cos(90° - A)$, $\cos A = \sin(90° - A)$, &c.

39. By means of the above formulæ we may express any Ratio of an angle in terms of any other Ratio.

Ex. Express all the trigonometrical Ratios in terms of the sine.

Ans. $\cos = \sqrt{(1-\sin^2)}$, $\tan = \dfrac{\sin}{\cos} = \dfrac{\sin}{\sqrt{(1-\sin^2)}}$, $\cot = \dfrac{1}{\tan} = \dfrac{\sqrt{(1-\sin^2)}}{\sin}$,

$$\sec = \frac{1}{\cos} = \frac{1}{\sqrt{(1-\sin^2)}},\ \text{cosec} = \frac{1}{\sin}.$$

40. The following will also be found a very useful result to be remembered.

If $\tan = \dfrac{a}{b}$, then $\sin = \dfrac{a}{\sqrt{(a^2+b^2)}}$, $\cos = \dfrac{b}{\sqrt{(a^2+b^2)}}$.

For $\sec^2 = 1 + \tan^2 = 1 + \dfrac{a^2}{b^2} = \dfrac{a^2+b^2}{b^2}$; $\therefore \cos = \dfrac{1}{\sec} = \dfrac{b}{\sqrt{(a^2+b^2)}}$;

and $\sin^2 = 1 - \cos^2 = 1 - \dfrac{b^2}{a^2+b^2} = \dfrac{a^2}{a^2+b^2}$; $\therefore \sin = \dfrac{a}{\sqrt{(a^2+b^2)}}$.

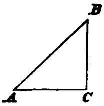
Ex. 1. Given tan = 2, find the sine and secant.

Here $\tan = \frac{2}{1}$; $\therefore \sin = \frac{2}{\sqrt{(4+1)}} = \frac{2}{\sqrt{5}}$, $\cos = \frac{1}{\sqrt{5}}$, $\sec = \frac{1}{\cos} = \sqrt{5}$.

Ex. 2. Since $\tan = \frac{\tan}{1}$, $\therefore \sin = \frac{\tan}{\sqrt{(1+\tan^2)}}$, $\cos = \frac{1}{\sqrt{(1+\tan^2)}}$.

41. *To find the sine, cosine, &c. of* 45°.

Let ABC be a triangle, right-angled at C, and having equal sides, CA, CB, and, therefore, also equal angles, CAB, CBA. Hence, since these two angles make up together 90°, each of them must be an angle of 45°.



Now $AB^2 = AC^2 + BC^2 = 2AC^2$ or $2BC^2$, that is,

$$AB = \sqrt{2}.AC \text{ or } \sqrt{2}.BC:$$

$$\therefore \sin 45^\circ = \frac{BC}{AB} = \frac{1}{\sqrt{2}} = \cos 45^\circ, \text{ and } \tan 45^\circ = \frac{BC}{AC} = 1;$$

$$\text{and} \therefore \text{ also } \cot 45^\circ = 1, \ \sec 45^\circ = \sqrt{2} = \operatorname{cosec} 45^\circ.$$

If we express the angle 45° in circular measure, these results may be written

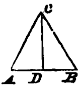
$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}, \ \tan\frac{\pi}{4} = 1 = \cot\frac{\pi}{4}, \ \sec\frac{\pi}{4} = \sqrt{2} = \operatorname{cosec}\frac{\pi}{4}.$$

Notice also that, if A be greater than 45°, and, therefore, greater than B, $BC > AC$, and $\sin A > \cos A$;

whereas, if A be less than 45°, $\cos A > \sin A$.

42. *To find the sine, cosine, &c. of* 30° *and* 60°.

Let ACB be an equilateral triangle, and, therefore, also equiangular: then, since its three angles make up together 180°, each of them must be an angle of 60°.



Draw CD perpendicular to AB, and, therefore, (Euc. I. 10) bisecting both the side AB and the angle ACB, so that $\angle ACD = 30^\circ$.

Then $AC = AB = 2AD$, and $CD^2 = AC^2 - AD^2 = 3AD^2$:

$$\therefore \sin 30^\circ = \frac{AD}{AC} = \frac{1}{2} = \cos 60^\circ,$$

$$\cos 30^\circ = \frac{CD}{AC} = \frac{\sqrt{3}.AD}{2AD} = \frac{\sqrt{3}}{2} = \sin 60^\circ,$$

$$\tan 30^\circ = \frac{AD}{CD} = \frac{1}{\sqrt{3}} = \cot 60^\circ;$$

whence $\cot 30^\circ = \sqrt{3} = \tan 60^\circ$, $\sec 30^\circ = \frac{2}{\sqrt{3}} = \operatorname{cosec} 60^\circ$,

$$\operatorname{cosec} 30^\circ = 2 = \sec 60^\circ.$$

It will be found sufficient to mark well the values of the *sine* and *cosine* of 30°, from which all the other Ratios, both of 30° and 60°, may be at once obtained:

$$\text{thus } \tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1}{\sqrt{3}}, \text{ \&c.}$$

If we use the circular measure, the above results become

$$\sin\frac{\pi}{6} = \frac{1}{2} = \cos\frac{\pi}{3},\ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3},\ \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \cot\frac{\pi}{3}, \text{ \&c.}$$

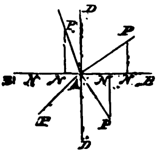
Ex. 5.

1. Express the sine in terms of the secant and cotangent.
2. Express the cosine in terms of the cosecant and cotangent.
3. Express the tangent in terms of the sine and cosecant.
4. Express the cotangent in terms of the cosine and secant.
5. Express the secant in terms of the cotangent, and the cosecant in terms of the secant.
6. Express the sine in terms of the versed sine.
7. Given tan = $\frac{3}{4}$, find all the other Ratios.
8. Given sin = $\frac{2}{3}$, find the cotangent and secant.
9. Given tan = $\frac{1}{2}$, find the cosecant and cosine.
10. Given sec = 4, find the sine and versed sine.
11. Shew that $\left(\sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right)\left(\sin\frac{\pi}{3} - \cos\frac{\pi}{3}\right) = \sin\frac{\pi}{6}$.
12. Shew that

$$\frac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ} = \cos 60^\circ, \text{ and } \frac{\sin 45^\circ - \sin 30^\circ}{\sin 45^\circ + \sin 30^\circ} = \frac{\sec 45^\circ - \tan 45^\circ}{\sec 45^\circ + \tan 45^\circ}.$$

43. We have hitherto been considering the Ratios of angles in the first quadrant only, or less than 90°: but the definitions of (32) hold equally for all angles whatever, as seen in the figure annexed.

Here BB', DD', are drawn, as usual, intersecting at right angles in A; the position of the revolving line is taken in the first, second, &c. quadrant respectively, and PN is drawn perpendicular on BB'. Now in each case we have, as before,



$$\sin A = \frac{PN}{AP}, \quad \cos A = \frac{AN}{AP}, \quad \tan A = \frac{PN}{AN}, \quad \&c.$$

Now, however, in accordance with (6), we shall consider PN to be positive in the first and second quadrants, where it is drawn *downwards* upon BB', and negative in the third and fourth quadrants, where it is drawn *upwards;* and so, likewise, we shall take AN to be positive in the first and fourth quadrants, where it is drawn to the *right* of A, and negative in the second and third quadrants, where it is drawn to the *left.* AP is in all cases to be positive, because it is always drawn from A *in the direction of the revolving line,* being measured off upon it.

N.B. In other cases of algebraic geometry, cases occur where AP is negative, being measured, not on that part of the revolving line which is actually describing the angle, but upon the line produced *backwards.* The Student, however, will perceive that the word 'direction' in (6) is not to be understood of a *fixed* direction only, as in the case of AN, or even of a direction *parallel to a fixed line,* as in that of PN, but of any *determinate* direction whatever, defined by the conditions under which the figure has to be drawn, as, in the case now before us, *along a revolving line.* And this instance will further illustrate the meaning of the words in (6), '*lines of the same kind*, &c.': for AP, we see, is *positive,* though it may lie in the direction AB' or AD', in which directions AN or PN respectively will be *negative*—the reason of course being that the Law, according to which AP is drawn, is *not* contradicted by its being drawn in the direction AB' or AD'.

44. Now, strictly, it would be most correct to denote the value of any of these lines by some algebraical letter, supposed to express the number of units of length it contains. Thus, for instance, we might use p, b, r for the *perpendicular* PN, the *base* AN, and the revolving line, or *radius*, AP, respectively; and then we should say that, in the first quadrant, $PN = +p$, $AN = +b$, in the second, $PN = +p$, $AN = -b$, in the third, $PN = -p$, $AN = -b$, in the fourth, $PN = -p$, $AN = +b$, while, in each quadrant, $AP = +r$.

But it is more usual, (as it is often more convenient,) to use the *geometrical* symbols PN, &c. themselves in an *algebraical* way, namely, so as to express by them, not the lines themselves as they actually stand in the figure, but merely their *positive* values; and thus, if the lines are drawn positively, their values will be expressed by $+PN$, $+AN$, or, if negatively, by $-PN$, $-AN$. Hence we shall no longer have $\sin A = \frac{PN}{AP}$ in *every* quadrant, but $\frac{+PN}{AP}$ or $\frac{-PN}{AP}$, that is $+\frac{PN}{AP}$ or $-\frac{PN}{AP}$, according to the position of PN: and so with the other Ratios.

45. *To trace the changes in sign of the different Ratios in the four quadrants.*

(1) $\sin A = \dfrac{+PN}{AP}$ in the first and second quadrants,

$= \dfrac{-PN}{AP}$ in the third and fourth;

hence the *sine* is positive in the *first* and *second* quadrants, but negative in the third and fourth.

Similarly for the *cosecant.*

(2) $\cos A = \dfrac{+AN}{AP}$ in the first and fourth quadrants,

$= \dfrac{-AN}{AP}$ in the second and third;

hence the *cosine* is positive in the *first* and *fourth* quadrants, but negative in the second and third.

Similarly for the *secant.*

(3) $\tan A = \dfrac{+PN}{+AN}$ in the first quadrant, $= \dfrac{+PN}{-AN}$ in the second, $= \dfrac{-PN}{-AN}$ in the third, $= \dfrac{-PN}{+AN}$ in the fourth;

hence the tangent is positive in the *first* and *third* quadrants, in which PN and AN have both the same sign, but negative in the second and fourth.

Similarly for the cotangent.

These results will be best remembered by observing the *sequences* of signs in the four quadrants, namely, for the *sine* and *cosecant*, + + − −, for the *cosine* and *secant*, + − − +, for the *tangent* and *cotangent*, + − + −.

It must be observed, however, that all the results obtained in (37), with the angle in the first quadrant, where the Ratios are all positive, will, by (6), hold equally good for all angles whatever, and might be separately proved, though with more complexity, for each particular case. So, likewise, in all future reasonings, we shall be justified in drawing the simplest general figure we can, that is, with the lines and angles concerned all positive, if possible, and assuming the result thus obtained to be universally true.

46. *To trace the changes in magnitude of the different Ratios through the four quadrants.*

Whenever the revolving line AP coincides with BB', (as it does at first, and again when it has described 180°,) then the perpendicular PN vanishes entirely, and the base AN becomes equal in magnitude to the hypothenuse AP: and so, likewise, whenever AP coincides with DD', then AN vanishes, and PN becomes equal in magnitude to AP. It is plain also that we need only follow the changes of magnitude of the different Ratios through the first quadrant, since they go through the same exactly in the other three quadrants, alternately increasing and decreasing.

Observe that we are not now considering the *signs* of the Ratios, which will have to be separately determined for each quadrant as in (45): in the first quadrant, however, we know they are all positive.

Now, in the first quadrant, or as A changes from 0° to 90°,

$\sin A \left(=\frac{PN}{AP}\right)$ changes from $\frac{0}{r}$ to $\frac{r}{r}$, that is, from 0 to 1;

$\cos A \left(=\frac{AN}{AP}\right)$ . . . . $\frac{r}{r}$ to $\frac{0}{r}$, ... 1 to 0;

$\tan A \left(=\frac{PN}{AN}\right)$ . . . . $\frac{0}{r}$ to $\frac{r}{0}$, ... 0 to ∞;

$\cot A\ (=\frac{AN}{PN})$ changes from $\frac{r}{0}$ to $\frac{0}{r}$, that is, from ∞ to 0

$\sec A\ (=\frac{AP}{AN})$ . . . . $\frac{r}{r}$ to $\frac{r}{0}$, ... 1 to ∞

$\operatorname{cosec} A\ (=\frac{AP}{PN})$ . . . . $\frac{r}{0}$ to $\frac{r}{r}$, ... ∞ to 1.

47. Hence for all the four quadrants we may fon the following scheme, to express the changes in sig and magnitude of the trigonometrical functions of a angle *A*, as the angle changes from 0° to 360°.

A	sin*A*	cos*A*	tan*A*	cot*A*	sec*A*	cosec
0° to 90°	0 to 1 (+)	1 to 0 (+)	0 to ∞ (+)	∞ to 0 (+)	1 to ∞ (+)	∞ to (+)
90° to 180°	1 to 0 (+)	0 to 1 (-)	∞ to 0 (-)	0 to ∞ (-)	∞ to 1 (-)	1 to (+)
180° to 270°	0 to 1 (-)	1 to 0 (-)	0 to ∞ (+)	∞ to 0 (+)	1 to ∞ (-)	∞ to (-)
270° to 360°	1 to 0 (-)	0 to 1 (+)	∞ to 0 (-)	0 to ∞ (-)	∞ to 1 (+)	1 to (-)

The above Table must be well studied, and may l best remembered, by remarking carefully the results the first line, on which all the rest depend.

48. It will be observed, that the values of the si and cosine range between 0 and ± 1, so that the *always* lie between + 1 and − 1; and those of t secant and cosecant range between + 1 and + ∞, a between − 1 and − ∞, so that these *never* lie betwe + 1 and − 1; while those of the tangent and cotange range between 0 and ± ∞, and, therefore, may be any magnitude whatever, positive or negative.

Notice also that each of the Ratios changes its sig *whenever its value* passes through *zero* or *infinity*.

In fact, the same Ratio may be considered as having at the same moment the value $+0$ and -0: thus $\cos 90^\circ = +0$ or -0, according as we consider D to be the end of the first quadrant, in which the cosine is positive, or the beginning of the second, in which it is negative; and the same may be said of the values $+\infty$ and $-\infty$; so that we have $\tan 90^\circ = +\infty$ or $-\infty$. The real meaning, however, of such statements, is this: at the point D, where the angle becomes 90°, there is actually no such thing as a cosine or tangent at all, for no triangle APN can be formed, and the definitions, therefore, cannot be applied. But, on either side of D, *however close to it*, there *are* values of the cosine and tangent, those of the cosine exceedingly small, and those of the tangent exceedingly great; and these, however small in the one case, or however great in the other, will be positive or negative on different sides of D.

49. Hence also the versed-sine ($= 1 - \cos$) changes, in the four quadrants, from $1 - 1$ to $1 - 0$, from $1 - 0$ to $1 - (-1)$, from $1 - (-1)$ to $1 - 0$, and from $1 - 0$ to $1 - 1$, or from 0 to 1, from 1 to 2, from 2 to 1, and from 1 to 0. The versed-sine, therefore, is always *positive*, and its values increase gradually from 0 to 2 in the first two quadrants, and then diminish from 2 to 0.

The following is a *Miscellaneous* Exercise upon all the foregoing portions of the Book.

Ex. 6.

1. Find the number of grades in the angle whose circular measure is $\frac{1}{2}$.

2. Shew that the areas of two circles, in which the circumference of the one is equal to an arc of 60° in the other, are as $1 : 36$.

3. The distances of Venus and the Earth from the Sun are as the numbers 7 and 10, and they revolve in 224 and 365 days respectively: compare their rates of motion.

4. Compare the angles subtended by an arc of $3\frac{1}{3}$ feet to a radius of $2\frac{1}{2}$ feet, and an arc of $2\frac{1}{3}$ feet to a radius of $3\frac{1}{2}$ feet; and express the first in degrees and the second in grades.

5. Shew that $\sin^2\frac{\pi}{6}$, $\sin^2\frac{\pi}{4}$, $\sin^2\frac{\pi}{3}$, $\sin^2\frac{\pi}{2}$, are as 1, 2, 3, 4.

6. Find $\tan A$ from the equation $\sin^2 A + 5\cos^2 A = 3$.

7. Given $\frac{2\sin A - \cos A}{2\sin A + 3\cos A} = \frac{2}{3}$, find the sine, cosine, and tangent of A.

8. Determine $\sin\theta$ and $\tan\theta$ from the equation $\tan\theta + 3\cot\theta = 4$.

9. If $\cos x = \frac{\cos A}{\sin C}$ and $\cos(90° - x) = \frac{\cos B}{\sin C}$, shew that

$$\sin^2 A + \sin^2 B + \sin^2 C = 2.$$

10. If $\tan^2 A + 4\sin^2 A = 6$, find the value of $\operatorname{vers} A$.

11. If $\tan\theta = \frac{a}{b}$, find the value of

$$\frac{a\sin\theta + b\cos\theta}{a\sin\theta - b\cos\theta} \text{ and } 1 - \frac{\sin^2\theta}{1+\cot\theta} - \frac{\cos^2\theta}{1+\tan\theta}.$$

12. If $\sin A = \sqrt{2}\sin B$, and $\tan A = \sqrt{3}\tan B$, determine A and B.

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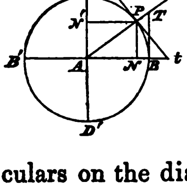
50. It was formerly customary to consider the trigonometrical functions as belonging to the subtending *arc* rather than to the *angle*, and to represent them by certain lines as follows.

Draw BB', DD' intersecting at right angles in A; and let BP be any arc of a circle, subtending at the centre A an angle of $A°$: then, as in (12), the angle DAP, or the arc DP, is the complement of $A°$, and the angle $B'AP$, or the arc $B'P$, is the supplement of $A°$. From P drop PN, PN', perpendiculars on the diameters through B and D; and let the



lines touching the circle at B and D meet the diameter through P produced in the points T, T'.

Then PN was called the *sine*, BT the *tangent*, and AT the *secant* of the arc BP or A; and PN', DT', AT', which, according to these definitions, would be the sine, tangent, and secant of the arc DP or $(90° - A)$, were called the *cosine, cotangent*, and *cosecant*, of A.

For PN', however, it is generally more convenient to use its equal AN, and for AT and AT', their equals At, At', found by producing the touching line at P to cut the diameters through B and D in t, t'.

So also BN was called the *versed-sine* of A, and DN', the versed sine of DP, was the *coversed-sine* of A: and likewise $B'N$, the versed sine of $B'P$, was called the *suversed-sine* of A.

51. Now, to radius *unity*, it will be seen that the above definitions exactly coincide with those already given of the trigonometrical Functions. For then each line in the figure will have for its numerical value, not the number of *feet* &c. it contains, but the number of times (integral or fractional) it contains the *radius*, that is, in other words, the ratio which it bears to the radius. Hence, to radius *unity*, the numerical values of the *lines* PN, AN, will be the same as those of the *ratios* $\frac{PN}{AP}$, $\frac{AN}{AP}$; and so, likewise, the values of the lines BT, AT, will be the same as those of the *ratios* $\frac{BT}{AB}$, $\frac{AT}{AB}$, or $\frac{PN}{AN}$, $\frac{AP}{AN}$, &c.

Similarly, $\text{vers}\, A = BN = AB - AN$,

and $\text{covers}\, A = DN' = AD - AN' = AD - PN$,

which become, when the lines are referred to the radius, or the radius is taken for *unity*, $1 - \cos A$ and $1 - \sin A$ respectively, as before. And, in like manner,

$$\text{suvers}\,A = B'N = B'A + AN = 1 + \cos A.$$

52. Hence may be seen the origin of the names given to the different Functions.

The name sine, from *sinus*, bosom, is the mere Latin translation of an Arabic word, PN being half the string, as it were, of the *arcus*, or bow, of which BP is half, and being the part brought up to the *breast* of the archer in discharging it. The names tangent (*touching* line) and secant (*cutting* line) explain themselves from the figure: and, of course, the cosine, cotangent, &c. were so called, as being the sine, tangent, &c. of the complement The name *versed-sine* is not so easy to explain: but BN was formerly called from its position the *sagitta*, or arrow, of the bow.

53. If, however, in any case, the radius be not taken as the unit of measurement, let r be its numerical value, expressed in terms of some other unit; then, if we use $\text{Sin}\,A$ (with a capital letter) to denote the *line* PN and $\sin A$ to denote the *ratio* $\frac{PN}{AP}$, we shall have

$$\sin A = \frac{\text{Sin}\,A}{r}, \quad \text{or} \quad \text{Sin}\,A = r \sin A;$$

and so with the other functions. Hence any result obtained for $\text{Sin}\,A$, $\text{Cos}\,A$, &c. to radius r may be adapted to the radius *unity*, or converted into the corresponding result for the Ratios, by writing $r \sin A$, $r \cos A$, &c. for $\text{Sin}\,A$, $\text{Cos}\,A$, &c.: and, conversely, any result, obtained by means of the Ratios, or by the lines to radius *unity*, may be converted into the corresponding result to the radius r, by writing

$$\frac{\text{Sin}\,A}{r}, \ \frac{\text{Cos}\,A}{r}, \ \text{\&c., for } \sin A, \ \cos A, \ \text{\&c.}$$

Ex. 1. $\sin^2 A + \cos^2 A = 1$ becomes, to radius r,

$$\frac{\text{Sin}^2 A}{r^2} + \frac{\text{Cos}^2 A}{r^2} = 1, \quad \text{or} \quad \text{Sin}^2 A + \text{Cos}^2 A = r^2.$$

Ex. 2. $\sin A = \dfrac{1}{\operatorname{cosec} A}$ becomes, to radius r,

$$\frac{\text{Sin}\, A}{r} = \frac{1}{\dfrac{\text{Cosec}\, A}{r}}, \quad \text{or} \quad \text{Sin}\, A = \frac{r^2}{\operatorname{cosec} A}.$$

It will be seen that this has, in fact, amounted to multiplying each term of the expression to radius *unity* by such a power of r, as will render the whole result *homogeneous*; it being observed that the quantities Sin A, Cos A, &c., being *lines* and not *ratios*, will be each, as well as r, of *one* dimension.

Ex. 3. If, to radius *unity*, we have $\cos 3A = 4\cos^3 A - 3\cos A$, then, to radius r, we shall have

$$\frac{\text{Cos}\, 3A}{r} = 4\frac{\text{Cos}^3 A}{r^3} - 3\frac{\text{Cos}\, A}{r}, \quad \text{or} \quad r^2\, \text{Cos}\, 3A = 4\, \text{Cos}^3 A - 3r^2\, \text{Cos} A;$$

the same result as we shall get, if we merely multiply each term of the given result by such a power of r, as shall make its dimensions equal to those of its highest term, which is plainly $\cos^3 A$.

Ex. 4. If, to radius *unity*, $2\sin A = \sqrt{(1 + \sin 2A)} + \sqrt{(1 - \sin 2A)}$, then, to radius r, $2\,\text{Sin}\, A = \sqrt{(r^2 + r\,\text{Sin}\, 2A)} + \sqrt{(r^2 - r\,\text{Sin}\, 2A)}$, where each term is made of one dimension.

The Student should *prove* the truth of this last adaptation.

54. According to the *line*-definitions then, it is plain that, unless the radius be taken for unity, the trigonometrical functions of any given angle are not fixed in value, as they are with the *ratio*-definitions, but are always, like the *arcs* themselves, *proportional to the radius* employed. It becomes necessary, therefore, to mention the radius in any such case, or, at least, to refer to it as understood. It is this inconvenience which has led to the general disuse of these lines.

Most commonly, however, the radius is understo
to be unity, and all the other lines are referred to
Here then the *numerical values* of the lines *PN*, &
will be the same for the same angle, whatever be the
actual length or the length of the radius, since th
will merely express the ratios which the lines be
to the radius, the value of the radius, or its ratio
itself, being 1. And this method of exhibiting t
trigonometrical Ratios will be found convenient
times, and may be used with advantage in the soluti
of some Problems.

Ex. 1. In the fig. of (50), $AP^2 = PN^2 + AN^2$, or $1 = \sin^2 A + \cos^2$

Ex. 2. $AT^2 = AB^2 + BT^2$, or $\sec^2 A = 1 + \tan^2 A$.

Ex. 3. Since $tt' = tP + Pt' = BT + DT' = \tan A + \cot A$,
$\therefore (\tan A + \cot A)^2 = (tt')^2 = At^2 + At'^2 = \sec^2 A + \operatorname{cosec}^2 A$.

55. It may be as well to give here the definitio
formerly used for the different trigonometrical fun
tions, referring to the figure in (50).

Let there be any arc *BP*, subtending an angle *BA*
to any radius: then if $BP = 90°$, *DP* is the compleme
of *BP*, and it should be observed, as in (12), th
whereas the arc *BP* begins at *B*, and increases in t
direction from *B* to *D*, the complement *DP*, on t
contrary, begins at *D*, and increases in the directi
from *D* to *B*. Now

(1) The *sine* (*PN*) of the arc or angle is the pe
pendicular drawn from the end of the arc upon t
diameter through the beginning of it;

(2) The *cosine* (*AN*) is that part of the sa
diameter, intercepted between the centre and the sin
and the *versed-sine* (*BN*) is that part of it, intercept
between the beginning of the arc and the sine;

(3) The *tangent* (BT) is the line drawn touching the circle at the beginning of the arc, and terminated by the diameter through the end of it produced;

(4) The *cotangent* (DT') is the line drawn touching the circle at the beginning of the complement, and terminated by the diameter through the end of it produced;

(5) The *secant* (AT or At) is the line drawn from the centre through either extremity of the arc, to meet the line which touches the circle at the other extremity;

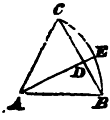
(6) The *cosecant* (AT' or At') is the line drawn from the centre through either extremity of the complement, to meet the line which touches the circle at the other extremity.

56. Let a line of unlimited length, revolving about B, from an initial position coincident with the touching line at B, always pass through the extremity of the arc BP: then the portion of this line, intercepted between the two extremities of the arc, that is, the line BP, is called the *chord* of the arc BP.

The above is a strictly correct definition of the chord, of which the use will be seen in (58): but, for all ordinary purposes, it will be sufficient to say that,

If BP be joined, then BP is the *chord* of the arc BP.

It is easily shewn that chord $A = 2\sin\frac{1}{2}A$.



For let BC be any arc A, and draw the radius ADE bisecting the chord BC (at right angles) in D, and its arc in E: then we have

$$BC = 2CD = 2\sin CE,$$

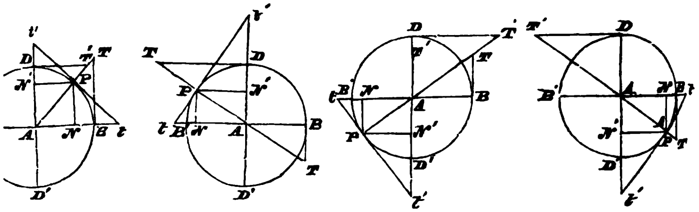
or chord $A = 2\sin\frac{1}{2}A$.

We may define the chord as a *Ratio*, as follows: Take *equal lengths* AB, AC, in the lines containing the angle BAC or A; then the ratio $\frac{BC}{AC}$, or $\frac{BC}{AB}$, is called the *chord* of A: and, as before,

$$\text{chord } A = \frac{BC}{AC} = \frac{2CD}{AC} = 2 \sin\tfrac{1}{2}A.$$

This Function, however, is not very frequently used.

57. We shall now trace the changes in sign and magnitude of the trigonometrical Functions, according to these *line*-definitions, and, as we shall find, with the same results as before.



The above figures will serve to shew the positions that will be taken by the different lines, drawn according to the definitions in (50), when the point P is in each of the four quadrants. As usual, the positions of the lines in the first quadrant are taken as the positive positions. The signs of the *sine* and *cosine* will be determined exactly as before by the signs of the lines PN and AN. The *versed-sine* BN, *coversed-sine* DN', and *suversed-sine* $B'N$, being always measured in the *same* directions from B, D, and B', will in each case be positive; the positive direction for the versed-sine being, however, the negative for the coversed-sine and suversed-sine. The tangent BT is positive in the first and third quadrant, and negative in the second and fourth. With respect to the *secant* AT, *it will be seen that*, in the second and third quadrants, the 'diameter through the end of the arc' (see Art. 55, Def. 3), in order to meet the line touching the circle at the beginning of the arc,' must be produced *backwards*, instead of *forwards*, as in the first and fourth quadrants, the secant being drawn in the direction from P towards A, instead of from A towards P: hence the secant is positive in the first and fourth quadrants, and negative in the second and third—a fact which we see yet more plainly by looking at the sign of At. The Student will easily be able to follow the signs of the *otangent* and *cosecant*, observing the directions of the *ositive* lines DT' and AT' or At'.

58. With respect to the chord, (found by joining BP,) we have seen (56) that strictly it is defined o be 'that portion of a line of unlimited length, revolving about B from an initial position coincident vith BT in fig. 1, which is intercepted between the eginning and end of the arc.' Now, when this line as revolved through 180°, it will become coincident vith BT in fig. 2, while the arc BP will have increased rom 0° to 360°, and hitherto the chord will have been neasured off upon the revolving line itself; but, after his, while the line still goes on revolving, and the arc BP increasing from 360° to 720°, the chord will have o be taken off upon the line produced *backwards*, until t has gone on revolving through another 180°, and becomes again coincident with BT. Hence the chord will e positive, while BP, or A, increases from 0° to 360°, nd negative, while A increases from 360° to 720°.

This appears also from the formula $\text{chord} A = 2 \sin \frac{1}{2} A$; or $\sin \frac{1}{2} A$ is positive, while $\frac{1}{2} A$ is in the first or second uadrant, that is, while $\frac{1}{2} A$ lies between 0° and 180°, r *A between 0° and 360°*; and negative, while $\frac{1}{2} A$ is in the third or fourth quadrant, that is, while $\frac{1}{2}A$ lies between 180° and 360°, or A between 360° and 720°.

59. So also with respect to magnitude.

For, referring to the figures in (57), it will be seen that, as the arc BP increases from 0° to 90°, the sine PN increases from 0 to AD, or (to radius *unity*) from 0 to 1, the cosine AN decreases from 1 to 0, the tangent BT increases continually from 0, and the secant AT from AB or 1, until, when the arc is near 90°, and AT becomes nearly coincident with AD, both BT and AT become exceedingly great, and *infinitely* great at last, when the arc is 90°, and AT actually coincides with AD; in which case AT becomes parallel to BT, and they never meet. On the other hand, when the arc begins at 0°, AT' is parallel to DT', and the cotangent and cosecant begin by being infinitely great, and then diminish continually as the arc increases, till, at 90°, the cotangent DT' vanishes, and the cosecant AT' becomes the radius AD or 1.

Thus the changes of magnitude are the same exactly in the first quadrant with the lines or with the Ratios; and in the other quadrants the same changes are repeated, the different functions alternately increasing and decreasing as before.

So also the versed-sine increases in the first two quadrants from 0 to BB', that is, from 0 to 2, and decreases from 2 to 0 in the other two quadrants: and, in like manner, we may reason for the coversed-sine and suversed-sine.

The chord increases in the first quadrant from 0 to BD, that is, from 0 to $\sqrt{2}$,

$$(\text{since } BD^2 = AB^2 + AD^2 = 1 + 1 = 2,)$$

in the second from $\surd 2$ to 2, in the third decreases from 2 to $\surd 2$, and in the fourth from $\surd 2$ to 0; after which, while the angle still increases between 360° and 720°, the chord (58) changes in sign, but goes through again the same changes in magnitude.

Ex. 7.

Prove by means of the formulæ in (37) the following formulæ, and adapt them to radius r:

1. $\sin A = \dfrac{\cos A}{\surd(\operatorname{cosec}^2 A - 1)}$. 2. $\operatorname{vers} A = \dfrac{\sec A - 1}{\sec A} = \dfrac{\operatorname{cosec} A - \cot A}{\operatorname{cosec} A}$.

3. $\sin A . \cos A = \dfrac{1}{\tan A + \cot A}$. 4. $\cot^2 A \cos^2 A = \cot^2 A - \cos^2 A$.

5. $\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A$.

6. $\tan A + \cot A = \surd(\sec^2 A + \operatorname{cosec}^2 A)$.

Prove by means of the line-definitions the following formulæ, and adapt them to radius r:

7. $\sin A = \dfrac{1}{\operatorname{cosec} A}$. 8. $\tan A \cot A = 1$. 9. $\tan A = \dfrac{\sin A}{\cos A}$.

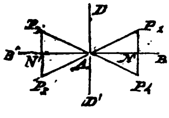
10. $\sin A = 2 \sin \frac{1}{2} A . \cos \frac{1}{2} A$. 11. $\cos A = 1 - 2 \sin^2 \frac{1}{2} A$.

12. $\operatorname{chd}^2 A = 2 \operatorname{vers} A$.

---

60. To express the Trigonometrical Ratios of $180° - A$, $180° + A$, and $-A$, in terms of A.

Draw BB', DD', cutting at right angles in A: make $\angle BAP_1 = \angle B'AP_2 = \angle B'AP_3 = \angle BAP_4 = A$: then we shall have the *positive* angles $BAP_1 = A$, $BAP_2 = 180° - A$, $BAP_3 = 180° + A$, $BAP_4 = 360° - A$, and the *negative* angles $BAP_4 = -A$, $BAP_3 = -(180° - A)$, &c.



Take $AP_1 = AP_2 = AP_3 = AP_4$; and draw P_1P_4, P_2P_3, cutting BB' at right angles in N, N'. Then the triangles AP_1N, AP_2N', AP_3N', AP_4N, are equal in all respects, so that $P_1N = P_2N' = P_3N' = P_4N$, and

$AN = AN'$. Hence it is manifest that any function o either of the above angles is equal in magnitude to th *same* function of A, whether differing, or not, in sign: thu

$$\sin(180^\circ - A) = \frac{+P_2N'}{AP_2} = +\frac{P_1N}{AP_1} = +\sin A,$$

$$\cos(180^\circ + A) = \frac{-AN'}{AP_3} = -\frac{AN}{AP_1} = -\cos A,$$

$$\tan(-A) = \frac{-P_4N}{+AN} = -\frac{P_1N}{AN} = -\tan A, \text{ \&c.}$$

The only difficulty then will be to know when th sign, which connects the two equal Ratios in any case is to be (+) or (–). And for this we observe that, sinc *all* the Ratios of A are *positive*, the sign to be use in any case must be that which belongs to the equa Ratio of $180^\circ - A$, $180^\circ + A$, or $-A$.

Hence, attending to the signs of the different Ratio in the four quadrants, as laid down in (45), we have

$$\sin(180^\circ - A) = +\sin A, \quad \sin(180^\circ + A) = -\sin A,$$
$$\sin(-A) = -\sin A;$$
$$\cos(180^\circ - A) = -\cos A, \quad \cos(180^\circ + A) = -\cos A,$$
$$\cos(-A) = +\cos A;$$
$$\tan(180^\circ - A) = -\tan A, \quad \tan(180^\circ + A) = +\tan A,$$
$$\tan(-A) = -\tan A:$$

and the cotangent, secant, and cosecant, follow the la of the tangent, cosine, and sine, respectively.

Hence also, as in (51),

$$\operatorname{suvers} A = \operatorname{vers}(180^\circ - A) = 1 - \cos(180^\circ - A) = 1 + \cos A$$

61. Of course, it follows that the converse of thes equations are true;

thus $\sin A = -\sin(-A)$, $\cos A = +\cos(-A)$.

It is plain also, from the figure, as observed in (7), that the Ratios of the angle BAP_4 will be the same, whether we regard it as the positive angle $(360° - A)$, or the negative angle $(-A)$: and, generally, that the Ratios of any angle will not be at all affected, by supposing the angle increased or diminished by any multiple of 360° (or 2π), since this only brings the revolving line to the same position exactly as before.

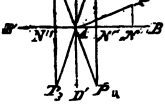
It is useful to notice the *pairs* of quadrants in which the different Ratios are the same, in sign as well as in magnitude, with the corresponding Ratios of A, namely, the sine and cosecant in the *first* and *second*, the cosine and secant in the *first* and *fourth*, the tangent and cotangent in the *first* and *third*,—in fact, the same pairs of quadrants as those in which (45) the respective Ratios are *positive*.

Of course we may write the above results, if we please, in circular measure: thus

$$\sin(\pi-\theta) = +\sin\theta, \quad \sin(\pi+\theta) = -\sin\theta, \quad \sin(-\theta) = -\sin\theta, \text{ \&c.}$$

62. In like manner, if we now make

$$\angle DAP_1 = \angle DAP_2 = \angle D'AP_3 = \angle D'AP_4 = \angle BAP = A,$$



then we shall have the positive angles

$$BAP_1 = 90° - A, \quad BAP_2 = 90° + A,$$
$$BAP_3 = 270° - A, \quad BAP_4 = 270° + A, \text{ \&c.},$$

and the negative angles

$$BAP_4 = -(90° - A), \quad BAP_3 = -(90° + A), \text{ \&c.}$$

Take, as before, $AP_1 = AP_2 = AP_3 = AP_4 = AP$, and draw PN, $P_1N'P_4$, $P_2N''P_3$, perpendiculars on BB'. Then the triangles AP_1N', &c. are, each of them, equal in all respects to APN, but so that $P_1N' = P_2N'' =$ &c. $= AN$, and $AN' = AN'' = PN$. Hence it is manifest that any function of either of the angles BAP$_1$, &c. will be equal in magnitude to the corresponding *cofunction* of A, whether differing, or not, in sign: thus

$$\sin(90^\circ - A) = \frac{+P_1N'}{AP_1} = +\frac{AN}{AP} = +\cos A,$$

$$\cos(90^\circ + A) = \frac{-AN''}{AP_2} = -\frac{PN}{AP} = -\sin A, \text{ \&c.}$$

So that, determining the signs as before, we have

$$\sin(90^\circ - A) = +\cos A, \quad \cos(90^\circ - A) = +\sin A,$$
$$\tan(90^\circ - A) = +\cot A, \text{ as in (35);}$$
$$\sin(90^\circ + A) = +\cos A, \quad \cos(90^\circ + A) = -\sin A,$$
$$\tan(90^\circ + A) = -\cot A; \text{ \&c.}$$

63. We may now shew, as stated in (34), that any function of any angle whatever may be expressed as a function of an angle less than 45°.

First leave out of the angle any multiple it may contain of 360° (or 2π); for by (61) the given function of the remaining angle will be the same as that of the original angle. Then put this remainder in the form $n.90^\circ \pm A$, which may always be done so as to have A less than 45°, by taking that multiple of 90° which is *nearest* to the angle in question, whether greater or less than it. And now, according as n is *even* or *odd*, the given function of $n.90^\circ \pm A$ may be written down at once as the corresponding *function* or *cofunction* of A, with that sign which belongs to the given function of $n.90^\circ \pm A$.

Ex. 1. $\sin 150^\circ = \sin(90^\circ + 60^\circ) = +\cos 60^\circ = \frac{1}{2}$:
here, however, we should most naturally have used the formula

$\sin A = \sin(180^\circ - A)$, or $\sin 150^\circ = \sin 30^\circ = \frac{1}{2}$, as before.

Ex. 2. $\cot 330^\circ = \cot(3.90^\circ + 60^\circ) = -\tan 60^\circ = -\sqrt{3}$;

or $= \cot(4.90^\circ - 30^\circ) = -\cot 30^\circ = -\sqrt{3}$, where $30^\circ < 45^\circ$.

Ex. 3. $\sin 369° = \sin 9°$, by merely leaving out $360°$;

$$\tan 500° = \tan 140° = \tan(180° - 40°) = -\tan 40°;$$

$$\cot 660° = \cot 300° = \cot(3.90° + 30°) = -\tan 30°;$$

$$\sec 820° = \sec 100° = \sec(90° + 10°) = -\operatorname{cosec} 10°;$$

$$\operatorname{cosec} 900° = \operatorname{cosec} 180° = \infty.$$

Ex. 8.

1. Write down the values of the cosine and cosecant of 150°, the sine and tangent of 135°, the tangent and secant of 120°.
2. Write down the values of the sine, cosine, and tangent of 240°.
3. Find the chord and versed sine of 120° and 300°.
4. Find the coversed sine of 330°, and the suversed sine of 315°.
5. Express $\sin 100°$, $\cos 200°$, $\tan 300°$, $\cot 400°$, $\sec 500°$, $\operatorname{cosec} 600°$, as functions of angles less than 45°.

Simplify the following expressions:

6. $1 + \{\sin(\frac{\pi}{2} - \theta) + \cos(\theta - \frac{\pi}{2})\} \times \{\cos(\frac{3\pi}{2} - \theta) + \sin(\theta - \frac{3\pi}{2})\}$.
7. $\sin(\pi+\theta)\cos(2\pi-\theta)+\cos(\frac{\pi}{2}+\theta)\sin(\frac{3\pi}{2}+\theta)+\tan(\pi+\theta)\tan(\frac{3\pi}{2}-\theta)$.
8. $\tan(A - 180°) + \cot(90° + A) + \tan(180° - A) + \cot(A - 270°) + \tan(180° + A) + \cot(270° - A)$.
9. Express $\sin\{(4n+1)\frac{\pi}{2} \pm \theta\}$ and $\tan\{(4n+3)\frac{\pi}{2} \pm \theta\}$ in terms of θ.
10. Shew that

$$\operatorname{vers}(90° + A)\operatorname{vers}(A - 90°) + \operatorname{covers}(270° - A)\operatorname{covers}(A - 270°) = 1.$$

11. Given $\sec(\pi - \theta)\cos(\frac{\pi}{2} + a)\cos(\pi - a) = \tan(\frac{\pi}{2} - a)\sin(\frac{3\pi}{2} + \theta)$, find $\sec\theta$.
12. Shew that the sum of the suversed-sine of the complement of A and the coversed-sine of the supplement of A is 2.

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64. *To obtain general expressions for all angles which have the same given function as* A.

(1) $\sin A = +\sin(180° - A)$, and $\operatorname{cosec} A = +\operatorname{cosec}(180° - A)$: hence by (61) all angles will have the same sine and cosecant as A, which are included in either of the expressions $n.360° + A$ or $n.360° + (180° - A)$, where $n.360°$ represents a multiple of 360°, n being any integer positive or negative, including *zero*.

Now these expressions may be written

$$2n.180^\circ + A \quad \text{and} \quad (2n+1)\,180^\circ - A,$$

where, we observe, the sign is + or −, according as the multiple of 180° is *even* or *odd:* hence they may be both included in the single formula, $n.180^\circ + (-1)^n A$, since $(-1)^n = +1$ or -1, according as n is even or odd.

(2) $\cos A = +\cos(-A)$, and $\sec A = +\sec(-A)$: hence all angles will have the same cosine and secant as A, which are included in either of the expressions

$$n.360^\circ + A \quad \text{or} \quad n.360^\circ - A,$$

or, in one formula, $2n.180^\circ \pm A$, the multiple of 180° being here always *even.*

(3) $\tan A = +\tan(180^\circ + A)$, and $\cot A = +\cot(180^\circ + A)$: hence all angles will have the same tangent and cotangent as A, which are included in either of the expressions $n.360^\circ + A$ or $n.360^\circ + (180^\circ + A)$, which may be written $2n.180^\circ + A$ or $(2n+1)\,180^\circ + A$, or, in one formula, $n.180^\circ + A$, where n may be even or odd, the sign being always +.

For the circular measure, the above results become

$$\sin\theta = \sin\{n\pi + (-1)^n\theta\}, \quad \cos\theta = \cos(2n\pi \pm \theta),$$
$$\tan\theta = \tan(n\pi + \theta);$$

with the corresponding expressions for the other Ratios.

Ex. Since $\sin 30^\circ = \frac{1}{2}$, therefore the general value of all angles, whose sine is $\frac{1}{2}$, will be $n\pi + (-1)^n \frac{\pi}{6} = \frac{6n + (-1)^n}{6}\pi$; from which expression, by giving n the values 0, 1, 2, &c. successively, we get the series of angles $\frac{1}{6}\pi$, $\frac{5}{6}\pi$, $\frac{13}{6}\pi$, &c.

65. It should be observed that $\sin A$, when determined from $\cos A$, has properly *two* values, namely,

$$+\sqrt{(1-\cos^2 A)} \quad \text{and} \quad -\sqrt{(1-\cos^2 A)}:$$

and this we may expect beforehand. For, when $\cos A$

is given, A itself is not given, but may be any of the angles expressed by the formula $2n\pi \pm a$, using a for the least of those angles which has the same given cosine. Hence we should expect that any formula, which we may be able to obtain for $\sin A$ in terms of $\cos A$, would give us also the general value of $\sin(2n\pi \pm a)$.

Now $\sin(2n\pi \pm a) = \pm \sin a$, and admits, therefore, of two equal values with different signs, which is just what we find when $\sin A$ is expressed in terms of $\cos A$. And so in other similar cases.

Ex. To shew that the sine, when determined from the tangent, will have two values.

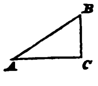
Here $\tan A$ is given to find $\sin A$; and $\tan A = \tan(n\pi + a)$; we may expect then that our result will give us the value of $\sin(n\pi + a)$, and this will be $\pm \sin a$, according as n is even or odd.

Thus, if $\tan A = \frac{3}{4}$, then (32 Ex.) $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$; but these should properly be $\pm \frac{3}{5}$, $\pm \frac{4}{5}$, since the den^r. is $\sqrt{(16 + 9)} = \pm 5$: and this we should expect; for, if $\tan A = \frac{3}{4}$, we have nothing to tell us whether A is to be an angle in the *first* or *third* quadrant, in either of which the tangent is positive, though the sine and cosine are both positive in the one, and both negative in the other.

So, if $\tan A = -\frac{3}{4}$, we shall have $\sin A = \pm \frac{3}{5}$, $\cos A = \mp \frac{4}{5}$.

66. The preceding results may now be practically employed, in certain simple cases, to find the heights, &c. of objects, as in the following Example.

Ex. A person, standing at a distance of 80 feet from the foot of a tower BC, observes the angle BAC, which a line from his eye to the summit of the tower makes with the horizontal line AC, to be 60°. Find the height of the tower.



Here, $BC = AC \tan A = 80 \tan 60° = 80\sqrt{3} = 80 \times 1.7320\&c = 138.56$ $= 139$ feet nearly, to which adding the height of the observer's eye at A, say 6 feet, we shall have the height of the tower 145 feet.

The angle BAC in the above is called the *angle of elevation*, or, simply, the *elevation* of the tower. And, in like manner, if an object at A were viewed from a higher position at B, the angle which the line BA makes with the horizontal line through the eye at B, or its equal, the angle BAC, is called the *angle of depression* of the object at A. Such angles would be commonly observed with an instrument, called a *Sextant*, held in the observer's hand.

Ex. 9.

1. Given $\tan A = 1$, write down the general value of A.
2. Find the general value of A, when $\tan 2A = \pm 1$.
3. Find the general value of A, when $\sec A = -2$.
4. Given $\cos 2A = \frac{1}{2}$, find the general value of A.
5. Given $\tan 5A = \sqrt{3}$, find the general value of A.
6. Shew that $\sec A$, in terms of $\tan A$, will have two values.
7. Shew that $\cos 2A$, in terms of $\sin A$ or $\cos A$, will have but one value; but $\sin 2A$, in terms of $\sin A$ or $\cos A$, two values.
8. A ladder, 30 feet long, inclined at an angle of 60°, rests against a wall; find the height of its top from the ground, and the distance of its foot from the base of the wall.
9. From the top of a ship's mast, 90 feet above the surface of the water, the angle of depression of the hull of another ship was found to be 30°: find the distance between the ships.
10. A May-pole being broken off by the wind, its top struck the ground at an angle of 45°, and at a distance of 21 feet from the foot of the pole: what was its whole height?
11. The shadow of a church-tower extends 56 yards from its base; find its height, it being observed that a two-foot rule, held vertically, casts a shadow of 4 ft 3 in.
12. A ladder, 45 feet long, being placed in a street, will exactly reach to a window 27 feet from the ground on one side: and upon being turned over without moving the foot, so as to lie at right angles to its former position, it will just reach a window on the other side of the street: determine the height of this latter window, and the breadth of the street.

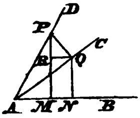
CHAPTER III.

ON THE TRIGONOMETRICAL RATIOS OF THE SUM AND DIFFERENCE, MULTIPLES AND SUBMULTIPLES, OF ANGLES.

67. *To find the sine and cosine of the sum and difference of two angles in terms of the angles themselves.*

Let $BAC\,(A)$ and $CAD\,(B)$ be two given angles: then, according as we measure CAD *forwards* or *backwards* from AC, the angle BAD will be expressed by $(A+B)$ or by $(A-B)$.

(1) To find the sine and cosine of $(A+B)$.



Take any point P in AD, and draw PM, PQ, perpendiculars on AB, AC, QN, QR, perpendiculars on AB, PM: then we have

$$\angle QPR = 90^\circ - \angle PQR = \angle RQA = \angle QAN = A.$$

Hence

$$\sin(A+B) = \sin BAP = \frac{PM}{AP} = \frac{QN+PR}{AP} = \frac{QN}{AP} + \frac{PR}{AP}$$

$$= \frac{QN}{AQ}\cdot\frac{AQ}{AP} + \frac{PR}{PQ}\cdot\frac{PQ}{AP}$$

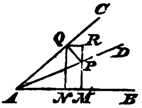
$$= \sin A.\cos B + \cos A.\sin B;$$

$$\cos(A+B) = \cos BAP = \frac{AM}{AP} = \frac{AN-QR}{AP} = \frac{AN}{AP} - \frac{QR}{AP}$$

$$= \frac{AN}{AQ}\cdot\frac{AQ}{AP} - \frac{QR}{PQ}\cdot\frac{PQ}{AP}$$

$$= \cos A.\cos B - \sin A.\sin \mathrm{B}.$$

(2) To find the sine and cosine of $(A - B)$.



Drawing the figure as before, exce that QR will in this case fall on PM pr duced backwards, we have $\angle QPR = 90^\circ - \angle PQR = \angle RQC = \angle BAC = \angle$

Hence $\sin(A - B) = \sin BAP = \dfrac{PM}{AP} = \dfrac{QN - PR}{AP}$

$$= \frac{QN}{AQ} \cdot \frac{AQ}{AP} - \frac{PR}{PQ} \cdot \frac{PQ}{AP} = \sin A . \cos B - \cos A . \sin B;$$

and $\cos(A - B) = \cos BAP = \dfrac{AM}{AP} = \dfrac{AN + QR}{AP}$

$$= \frac{AN}{AQ} \cdot \frac{AQ}{AP} + \frac{QR}{PQ} \cdot \frac{PQ}{AP} = \cos A . \cos B + \sin A . \sin B.$$

68. We have given in (67) the complete *geometric* proof of the two cases: but the results of the first inclu those of the second by merely writing $-B$ for $+B$; thu

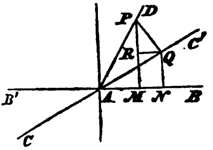
$$\sin(A - B) = \sin\{A + (-B)\} = \sin A . \cos(-B) + \cos A . \sin(-$$
$$= \sin A . \cos B - \cos A . \sin B, \quad .$$
$$\cos(A - B) = \cos\{A + (-B)\} = \cos A . \cos(-B) - \sin A . \sin(-$$
$$= \cos A . \cos B + \sin A . \sin B.$$

And these results we might have anticipated from th figure itself, since in the latter case the angle CA and the lines PR, QR, are all *negative*, being draw in opposite directions, with respect to the line AC ar the points P, Q, to those which in (1) we assume to be positive. Hence, looking at the results for (1 we should expect to get for (2), (as we actually do,)

$$\sin(A - B) = \frac{QN - PR}{AP} = \&c, \quad \cos(A - B) = \frac{AN + QR}{AP} = \&$$

In short, we have drawn the first figure in (67) for th *most simple* case; but, by the principle laid down in (6 *the results* are applicable to all cases whatsoever.

69. We shall here, however, give the strict geometrical Proof in one other case, in order to shew how it may be conducted in similar instances.



Suppose then that A lies between 180° and 225°, and $A - B$ between 45° and 90°: to find the sine and cosine of $A - B$.

The figure will be as annexed, where $BAC = A$, $CAD = B$, and PQ falls on AC produced backwards: hence we have

$$\sin(A - B) = \sin BAD$$

$$= \frac{PM}{AP} = \frac{QN + PR}{AP} = \frac{QN}{AQ}\cdot\frac{AQ}{AP} + \frac{PR}{PQ}\cdot\frac{PQ}{AP};$$

$$\cos(A - B) = \cos BAD$$

$$= \frac{AM}{AP} = \frac{AN - QR}{AP} = \frac{AN}{AQ}\cdot\frac{AQ}{AP} - \frac{QR}{PQ}\cdot\frac{PQ}{AP}.$$

But $\frac{QN}{AQ} = \sin BAC' = \sin(A - 180°) = -\sin(180° - A) = -\sin A$,

$$\frac{AN}{AQ} = \cos BAC' = \cos(A - 180°) = +\cos(180° - A) = -\cos A;$$

$$\frac{AQ}{AP} = \cos DAC' = \cos(180° - B) = -\cos B, \qquad \frac{PQ}{AP} = +\sin B;$$

$$\frac{PR}{PQ} = \sin PQR = \cos RQA = \cos BAC' = -\cos A, \quad \frac{QR}{PQ} = -\sin A:$$

hence $\sin(A - B) = (-\sin A)(-\cos B) + (-\cos A)(+\sin B)$

$$= \sin A \cos B - \cos A \sin B;$$

$$\cos(A - B) = (-\cos A)(-\cos B) - (-\sin A)(+\sin B)$$

$$= \cos A \cos B + \sin A \sin B.$$

70. So too we may derive $\cos(A+B)$ from $\sin(A+B)$; for $\cos(A+B) = \sin\{90° - (A+B)\} = \sin\{(90° - A) + (-B)\}$

$$= \sin(90° - A)\cos(-B) + \cos(90° - A)\sin(-B)$$

$$= \cos A.\cos B - \sin A.\sin B:$$

and, generally, in the same way, from any one of the four formulæ the others may all be deduced.

71. *To express* tan (A ± B) *in terms of* A *and* B.

$$\tan(A \pm B) = \frac{\sin(A \pm B)}{\cos(A \pm B)} = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B}$$

$$= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B},$$

by merely dividing every term by $\cos A . \cos B$.

Ex. 1. $\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 3$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \cos 15°;$$

so also $\sin 15° = \sin(45° - 30°) = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°;$

whence $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15°$, and $\tan 15° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75$

Ex. 2. $\tan(45° \pm A) = \dfrac{\tan 45° \pm \tan A}{1 \mp \tan 45° \tan A} = \dfrac{1 \pm \tan A}{1 \mp \tan A}$,
since $\tan 45° = 1$.

Ex. 10.

Prove the following results:

1. $\cos(60°-A) = \frac{1}{2}(\cos A + \sqrt{3} \sin A)$; $\sin(45°+A) = \frac{1}{\sqrt{2}}(\sin A + \cos$
2. $\text{vers}(A + 30°) - \text{vers}(A - 30°) = \sin A$.
3. $\dfrac{\sin(45° + A) - \cos(45° + A)}{\sin(45° + A) + \cos(45° + A)} = \tan A$.
4. $\dfrac{\tan(45° + A) - \tan(45° - A)}{\tan(45° + A) + \tan(45° - A)} = 2 \sin A \cos A$.
5. Shew that $\sin(n + 1)A = 2 \sin nA \cos A - \sin(n - 1)A$,
   and $\cos(n + 1)A = 2 \cos nA \cos A - \cos(n - 1)A$;
   and hence express $\sin 2A$, $\cos 2A$, in terms of $\sin A$, $\cos A$.
6. Express $\cot(A \pm B)$ in terms of $\cot A$ and $\cot B$.
7. Express $\sec(A \pm B)$ in terms of $\sec A$ and $\sec B$.
8. Express $\text{cosec}(A \pm B)$ in terms of $\text{cosec} A$ and $\text{cosec} B$.
9. Given the expression for $\sin(A-B)$ deduce that for $\cos(A-$
10. Given the expression for $\cos(A+B)$ deduce that for $\sin(A+$
11. Find (geometrically) the sine and cosine of $A - B$, when and $A - B$ are each between 90° and 180°.
12. Find the sine and cosine of $A + B$, when A lies betwe 90° and 180°, and $A + B$ between 180° and 270°.

72. *To express the ratios of* 2A *in terms of* A.

If in the formulæ for $\sin(A+B)$ and $\cos(A+B)$ we write A for B, we get immediately

$$\sin 2A = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A,$$
$$\cos 2A = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A.$$

For the latter formula, however, it is often convenient to use one of its equivalents; viz.

$$\cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1,$$
$$\text{or} \quad (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A.$$

So also, writing A for B in the expression for $\tan(A+B)$, we get

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2\tan A}{1 - \tan^2 A}.$$

73. We may derive other results, of course, from the above, by giving different values to A.

Thus, writing $\frac{1}{2}A$ for A, we get $\sin A = 2\sin\frac{1}{2}A \cos\frac{1}{2}A$,

$$\cos A = \cos^2\tfrac{1}{2}A - \sin^2\tfrac{1}{2}A = 2\cos^2\tfrac{1}{2}A - 1 = 1 - 2\sin^2\tfrac{1}{2}A; \text{ \&c.}$$

Hence $\dfrac{1 \pm \sin A}{\cos A} = \dfrac{(\cos^2\frac{1}{2}A + \sin^2\frac{1}{2}A) \pm 2\sin\frac{1}{2}A\cos\frac{1}{2}A}{\cos^2\frac{1}{2}A - \sin^2\frac{1}{2}A}$

$$= \frac{(\cos\frac{1}{2}A \pm \sin\frac{1}{2}A)^2}{\cos^2\frac{1}{2}A - \sin^2\frac{1}{2}A} = \frac{\cos\frac{1}{2}A \pm \sin\frac{1}{2}A}{\cos\frac{1}{2}A \mp \sin\frac{1}{2}A} = \frac{1 \pm \tan\frac{1}{2}A}{1 \mp \tan\frac{1}{2}A} = \tan(45^\circ \pm \tfrac{1}{2}A).$$

The step in the above by which $1 \pm \sin A$ is changed to $(\cos\frac{1}{2}A \pm \sin\frac{1}{2}A)^2$ should be particularly noticed, as it will be found to be often of great use in the solution of Problems.

74. Notice also the following expressions for $\tan A$.

$$\tan A = \frac{\sin A}{\cos A} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin 2A}{1 + \cos 2A},$$

$$\text{or} = \frac{2\sin^2 A}{2\sin A\cos A} = \frac{1 - \cos 2A}{\sin 2A}:$$

and so, $\tan\frac{1}{2}A = \dfrac{\sin A}{1 + \cos A}$, or $= \dfrac{1 - \cos A}{\sin A}$.

75. Hence also we may get $\sin 3A$, $\cos 3A$, $\tan 3A$, in terms of $\sin A$, $\cos A$, $\tan A$, respectively.

For $\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$
$= (2 \sin A \cos A) \cos A + (1 - 2\sin^2 A) \sin A$
$= 2 \sin A (1 - \sin^2 A) + (1 - 2 \sin^2 A) \sin A = 3 \sin A - 4 \sin^3 A$.

$\cos 3A = \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$
$= (2\cos^2 A - 1) \cos A - 2(1 - \cos^2 A) \cos A = 4\cos^3 A - 3\cos A$.

$\tan 3A = \tan(2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{\dfrac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1 - \tan^2 A}\tan A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Hence also, as before, by writing $\frac{1}{3}A$ for A, we get
$\sin A = 3\sin\frac{1}{3}A - 4\sin^3\frac{1}{3}A$, $\cos A = 4\cos^3\frac{1}{3}A - 3\cos\frac{1}{3}A$, &c.

76. *To express* sin A *and* cos A *in terms of* cos 2A.

By (72) $2\sin^2 A = 1 - \cos 2A$, $2\cos^2 A = 1 + \cos 2A$;

so that
$$\sin A = \sqrt{\{\tfrac{1}{2}(1 - \cos 2A)\}} = \tfrac{1}{2}\sqrt{(2 - 2\cos 2A)},$$
$$\cos A = \sqrt{\{\tfrac{1}{2}(1 + \cos 2A)\}} = \tfrac{1}{2}\sqrt{(2 + 2\cos 2A)}.$$

Hence
$$\tan^2 A = \frac{\sin^2 A}{\cos^2 A} = \frac{1 - \cos 2A}{1 + \cos 2A}, \text{ whence } \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A};$$
which last result, however, may be obtained more directly as follows:
$$\cos 2A = \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A},$$
if we divide every term by $\cos^2 A$.

Hence also, as before,
$$\left.\begin{aligned} 2\sin^2\tfrac{1}{2}A &= 1 - \cos A \\ 2\cos^2\tfrac{1}{2}A &= 1 + \cos A \end{aligned}\right\}, \quad \tan^2\frac{A}{2} = \frac{1 - \cos A}{1 + \cos A}, \quad \cos A = \frac{1 - \tan^2\frac{1}{2}A}{1 + \tan^2\frac{1}{2}A}.$$

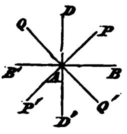
77. *To express* $\sin A$ *and* $\cos A$ *in terms of* $\sin 2A$.

In order to this, we might square the equation
$$\sin 2A = 2\sin A \cos A = 2\sin A\sqrt{(1 - \sin^2 A)},$$

and then we should get a biquadratic for determining $\sin A$ in terms of $\sin 2A$: but the following is a more simple method of arriving at the same result.

Since $(\sin A \pm \cos A)^2 = \sin^2 A + \cos^2 A \pm 2 \sin A \cos A$
$$= 1 \pm \sin 2A,$$
$\therefore \sin A + \cos A = \pm \sqrt{(1 + \sin 2A)}$, $\sin A - \cos A = \pm \sqrt{(1 - \sin 2A)}$: whence, adding and subtracting these equations, we get
$$2 \sin A = \pm \sqrt{(1 + \sin 2A)} \pm \sqrt{(1 - \sin 2A)},$$
$$2 \cos A = \pm \sqrt{(1 + \sin 2A)} \mp \sqrt{(1 - \sin 2A)}.$$

78. In the above results it is seen that the signs of the square-roots are not determined: nor can they be, until we know the value of A or of $2A$ in any case. We now proceed to fix them for different values of A.



Draw, as usual, BB', DD', intersecting at right angles in A; and draw also PP', QQ', so as to bisect these angles, and, therefore, making angles of 45° with BB' or DD'; these we shall call the *octant-lines*. Measuring, as before, all angles from AB, it is plain that for any angle, whose bounding line lies any where in either of the spaces, PAQ', QAP', that is, between BB' and either of the octant-lines, AN will be greater than PN, or the cosine will as to *magnitude* (without regard to *sign*) be greater than the sine; whereas for any angle, whose bounding line lies between DD' and either of the octant-lines, PN will be greater than AN, or the sine will be greater than the cosine.

Now $\sin A + \cos A$ is *positive*,

(1) when sin is (+) and cos (+),

that is, in the whole of the first quadrant;

or (2) when sin is (+) and cos (−), if sin > cos,

that is, in the *former* half of the second quadrant;

or (3) when sin is (−) and cos (+), if cos > sin,

that is, in the *latter* half of the fourth quadrant:

collecting which results we may say that

$$\sin A + \cos A = +\sqrt{(1 + \sin 2A)},$$

when A is an angle in the *first* quadrant, or in either of the two adjacent half-quadrants.

Similarly, $\sin A + \cos A = -\sqrt{(1 + \sin 2A)}$,
when A is an angle in the *third* quadrant, or in either of the two adjacent half-quadrants.

Again, $\sin A - \cos A$ is positive,

(1) when sin is (+) and cos (–),
that is, in the whole of the second quadrant;

or (2) when sin is (+) and cos (+), if sin > cos,
that is, in the *latter* half of the first quadrant;

or (3) when sin is (–) and cos (–), if cos > sin,
that is, in the *former* half of the third quadrant:
collecting which results we may say that

$$\sin A - \cos A = +\sqrt{(1 - \sin 2A)},$$

when A is an angle in the *second* quadrant, or in either of the two adjacent half-quadrants.

Similarly, $\sin A - \cos A = -\sqrt{(1 - \sin 2A)}$,
when A is an angle in the *fourth* quadrant, or in either of the two adjacent half-quadrants.

It will be seen that each of these formulæ holds good for 180° together, between alternate octant-lines, and that any *pair* of them, (namely, one for $\sin A + \cos A$, and one for $\sin A - \cos A$,) holds good for 90° together, between successive lines of octants: that is, for instance, all angles, whose bounding lines lie between AP and AQ will have the *same* pair of formulæ, namely,

$$\sin A + \cos A = +\sqrt{(1 + \sin 2A)}, \quad \sin A - \cos A = +\sqrt{(1 - \sin 2A)}.$$

When, therefore, any value of A is given, and we are required to find for it the corresponding pair of formulæ, we must consider in what quadrant the given angle is, and select the proper signs accordingly.

Ex. If $A = 30°$, then
$$\sin A + \cos A = +\sqrt{(1 + \sin 2A)}$$
$$\sin A - \cos A = -\sqrt{(1 - \sin 2A)};$$
$$\therefore \sin A = \tfrac{1}{2}\{\sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)}\},$$
$$\cos A = \tfrac{1}{2}\{\sqrt{(1 + \sin 2A)} + \sqrt{(1 - \sin 2A)}\}.$$

We may prove the truth of this result by trial: for $\sin 2A = \frac{1}{2}\sqrt{3}$;

$$\text{and } \therefore \sin A = \frac{1}{2}\{\sqrt{(1 + \frac{\sqrt{3}}{2})} - \sqrt{(1 - \frac{\sqrt{3}}{2})}\} = \frac{1}{2}\{\sqrt{\frac{4 + 2\sqrt{3}}{4}} - \sqrt{\frac{4 - 2\sqrt{3}}{4}}\}$$
$$= \frac{1}{2}\{\frac{\sqrt{3}+1}{2} - \frac{\sqrt{3}-1}{2}\} = \frac{1}{2};$$
$$\text{and } \cos A = \frac{1}{2}\{\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}\} = \frac{\sqrt{3}}{2}, \text{ as it should be.}$$

79. The reason why in (77) the expressions for $2 \sin A$ and $2 \cos A$ appear with so many undetermined signs is this, that, as in (65), we cannot help including in the steps, by which we arrive at the result, a great number of angles, which agree in certain of their trigonometrical properties, those, namely, expressed in the formulæ with which we start, but not in *all*, nor, in fact, in those for which we are seeking expressions. Thus there is a multitude of angles, for which the equation $(\sin A + \cos A)^2 = 1 + \sin 2A$ is true; but these angles may differ in their expressions for $\sin A + \cos A$, which in some of them may $= +\sqrt{(1 + \sin 2A)}$, and in others may $= -\sqrt{(1 + \sin 2A)}$.

Ex. II.

Prove the following formulæ:

1. $\cos A = \cos^4 \frac{1}{2}A - \sin^4 \frac{1}{2}A$.
2. $\cot A - \tan A = 2 \cot 2A$.
3. $\tan \frac{1}{2}A = \dfrac{\tan A}{1 + \sec A}$.
4. $\operatorname{cosec} 2A = \dfrac{1 + \cot^2 A}{2 \cot A}$.
5. $\sec A = 1 + \tan A \tan \frac{1}{2}A$.
6. $\operatorname{cosec} A = \frac{1}{2}(\tan \frac{1}{2}A + \cot \frac{1}{2}A)$.

7. $\dfrac{2\sin A + \sin 2A}{2\sin A - \sin 2A} = \cot^2 \tfrac{1}{2}A.$ 8. $\dfrac{\sin 2A}{1+\cos 2A} \times \dfrac{\cos A}{1+\cos A} = \tan\tfrac{1}{2}A.$

9. $\dfrac{1+\sin A}{1+\cos A} = \tfrac{1}{2}(1+\tan\tfrac{1}{2}A)^2.$ 10. $\dfrac{\operatorname{cosec} 2A}{1+\operatorname{cosec} 2A} = \dfrac{1+\tan^2 A}{(1+\tan A)^2}.$

11. $\cot A - \cot 2A = \operatorname{cosec} 2A.$ 12. $\tan\tfrac{1}{2}A + \sec\tfrac{1}{2}A = \tan(45^\circ + \tfrac{1}{4}A).$

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80. The formulæ of (67) may be combined in various ways, so as to produce other formulæ, which are of great use in trigonometrical operations.

Thus we have given

$$\sin(A+B) = \sin A\cos B + \cos A\sin B,$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B,$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B,$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

Hence $\sin(A+B) + \sin(A-B) = 2\sin A\cos B,$

$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B,$$
$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B,$$
$$\cos(A+B) - \cos(A-B) = -2\sin A\sin B,$$

or $\cos(A-B) - \cos(A+B) = 2\sin A\sin B.$

Again, $\sin(A+B)\times\sin(A-B)$

$$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$$
$$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B = \sin^2 A - \sin^2 B;$$

or $= (1-\cos^2 A)\cos^2 B - \cos^2 A(1-\cos^2 B) = \cos^2 B - \cos^2 A:$

$$\cos(A+B)\times\cos(A-B)$$
$$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$$
$$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B = \cos^2 A - \sin^2 B;$$

or $= (1-\sin^2 A)\cos^2 B - \sin^2 A(1-\cos^2 B) = \cos^2 B - \sin^2 A.$

The preceding results are very useful, and may be easily remembered by observing that each product is expressed by the difference of the squares of two functions of A and B, those functions being taken, the *first* out of the *first* term of the expressions for the factors multiplied, and the *second* out of the *second* term. Thus $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$, where $\sin A$ is taken out of $\sin A \cos B$, the *first* term of the expressions for $\sin(A \pm B)$, and $\sin B$ out of $\cos A \sin B$, the *second* term: but this product also $= \cos^2 B - \cos^2 A$, where $\cos B$ is taken out of the first term, and $\cos A$ out of the second.

Of course A and B in the above may stand for any angles whatever, as in the following Examples.

Ex. 1. $\sin(2A + 3B) + \sin(2A - 3B) = 2\sin 2A \cos 3B$.

Ex. 2. $2\sin(A + B)\cos(A - B)$

$$= \sin\{(A+B) + (A-B)\} + \sin\{(A+B) - (A-B)\}$$
$$= \sin 2A + \sin 2B.$$

Ex. 3. $\cos(A-B)\cos(B-C) = \frac{1}{2}\{\cos(A-B+B-C) + \cos(A-B-B+C)\}$

$$= \tfrac{1}{2}\{\cos(A-C) + \cos(A-2B+C)\}.$$

Ex. 4. $\cos(A + 30^\circ)\cos(30^\circ - A)$

$$= \cos(A+30^\circ)\cos(A-30^\circ) = \cos^2 A - \sin^2 30^\circ$$
$$= \cos^2 A - \tfrac{1}{4} = \tfrac{1}{2}(1+\cos 2A) - \tfrac{1}{4} = \tfrac{1}{4}(1 + 2\cos 2A).$$

Ex. 12.

Prove the following formulæ:

1. $\sin(30^\circ + A) + \sin(30^\circ - A) = \cos A$.
2. $\cos(30^\circ - \frac{1}{2}A) - \cos(30^\circ + \frac{1}{2}A) = \sin\frac{1}{2}A$.
3. $\sin(45^\circ + A)\sin(45^\circ - A) = \frac{1}{2}\cos 2A$.
4. $\cos(60^\circ + A)\cos(60^\circ - A) = \frac{1}{4}(2\cos 2A - 1)$.
5. $1 + \cos 2(A+B)\cos 2(A-B) = \cos^2 2A + \cos^2 2B$.
6. $2\sin\frac{1}{2}(A+B)\sin\frac{3}{2}(A-B) = \cos(A-2B) - \cos(2A-B)$.
7. $4\sin A\sin(60^\circ + A)\sin(60^\circ - A) = \sin 3A$.
8. $\sin(A+B)\sin(A-B) + \sin(B+C)\sin(B-C)$ $+ \sin(C+A)\sin(C-A) = 0$.
9. $\cos(A+2B)\cos(A-2B) + \sin(2B+C)\sin(2B-C)$ $= \cos(A+C)\cos(A-C)$.
10. $\sin(A-B)\sin C + \sin(B-C)\sin A + \sin(C-A)\sin B = 0$.
11. $\sin A\cos(A+B) - \cos A\sin(A-B) = \cos 2A\sin B$.
12. $\text{vers}(A+B)\,\text{vers}(A-B) = (\cos A - \cos B)^2$.

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81. All the preceding formulæ, in which functions of $A+B$ and $A-B$ are expressed in terms of A and B, may now be used to lead to others, in which functions of A and B are expressed in terms of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$, that is, of the semi-sum and semi-difference of A and B.

For $A = \frac{1}{2}(A+B) + \frac{1}{2}(A-B) = \alpha + \beta$, suppose,
and $B = \frac{1}{2}(A+B) - \frac{1}{2}(A-B) = \alpha - \beta$:

$$\therefore \sin A + \sin B = \sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$
$$= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B);$$
$$\sin A - \sin B = \sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$
$$= 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B);$$
$$\cos A + \cos B = \cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$
$$= 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B);$$
$$\cos B - \cos A = \cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$
$$= 2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

So also $\sin A\,\sin B = \sin(\alpha+\beta)\sin(\alpha-\beta)$

$$= \sin^2\alpha - \sin^2\beta = \sin^2\tfrac{1}{2}(A+B) - \sin^2\tfrac{1}{2}(A-B$$

or $$= \cos^2\beta - \cos^2\alpha = \cos^2\tfrac{1}{2}(A-B) - \cos^2\tfrac{1}{2}(A+B$$

$$\cos A\,\cos B = \cos(\alpha+\beta)\cos(\alpha-\beta)$$
$$= \cos^2\alpha - \sin^2\beta = \cos^2\tfrac{1}{2}(A+B) - \sin^2\tfrac{1}{2}(A-B$$

or $$= \cos^2\beta - \sin^2\alpha = \cos^2\tfrac{1}{2}(A-B) - \sin^2\tfrac{1}{2}(A+B$$

Hence also we get

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

with other similar results.

82. From the above it will be evident that *an formula whatever,* obtained for functions of $A+B$ an $A-B$ in terms of A and B, may be converted at on into a corresponding formula for functions of A and B i terms of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$, by merely writing in

A for $A+B$, and B for $A-B$,
$\frac{1}{2}(A+B)$ for A, and $\frac{1}{2}(A+B)$ for B.

Thus $\sin(A+B) = \sin A\cos B + \cos A\sin B$;

$$\therefore \sin A = \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + \cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B$$

which may be proved as before, by putting

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B) = \alpha+\beta,\quad B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B) = \alpha-\beta$$

$$\therefore \sin A = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \&c.$$

83. Since

$$\sin A \sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\},$$
$$\text{or} \quad = \sin^2\tfrac{1}{2}(A+B) - \sin^2\tfrac{1}{2}(A-B),$$

and

$$\cos A \cos B = \tfrac{1}{2}\{\cos(A+B) + \cos(A-B)\},$$
$$\text{or} \quad = \cos^2\tfrac{1}{2}(A+B) - \sin^2\tfrac{1}{2}(A-B),$$

we see that we may always resolve a product of two sines or two cosines, in two different ways, namely, either by means of a sum or difference, or by means of the difference of two squares; and the same is also true of the product of a sine and cosine, since

$$\sin A \cos B = \tfrac{1}{2}\{\sin(A+B) + \sin(A-B)\},$$

and also

$$= \sin A \sin(90^\circ - B) = \sin^2\tfrac{1}{2}(A+90^\circ-B) - \sin^2\tfrac{1}{2}(A-90^\circ+B)$$
$$= \sin^2\{45^\circ + \tfrac{1}{2}(A-B)\} - \sin^2\{45^\circ - \tfrac{1}{2}(A+B)\}.$$

Ex. 13.

Express (i) by means of a sum or difference, (ii) by means of the difference of two squares, the following products:

1. $\sin 5A \sin A$; $\sin 3A \sin 2A$; $\sin A \sin\frac{1}{2}A$.
2. $\cos 3A \cos\frac{1}{2}A$; $\cos 2A \cos A$; $\cos\frac{1}{2}A \cos\frac{1}{4}A$.
3. $\sin 2A \cos 3A$; $\sin\frac{1}{2}A \cos\frac{5}{2}A$; $\cos\frac{3}{2}A \sin A$.
4. $\sin(A+B)\sin C$; $\cos(A-B)\cos C$; $\sin(A+B)\cos(A-B)$.

Prove the following formulæ:

5. $\cos A + \cos(A+2B) = 2\cos B \cos(A+B)$.
6. $\cos^2\frac{1}{2}A\,(1+\sec\frac{1}{2}A+\tan\frac{1}{2}A)(1-\sec\frac{1}{2}A+\tan\frac{1}{2}A) = \sin A$.
7. $\tan\frac{3}{2}A \tan\frac{1}{2}A = \dfrac{\cos A - \cos 2A}{\cos A + \cos 2A}$.
8. $\tan^2 2A - \tan^2 A = \dfrac{\sin 3A \sin A}{\cos^2 2A \cos^2 A}$.
9. $\tan 2A - \tan A = \dfrac{2\sin A}{\cos A + \cos 3A}$.
10. $\tan A = \dfrac{\sin\frac{2}{3}A + \sin\frac{4}{3}A}{\cos\frac{2}{3}A + \cos\frac{4}{3}A}$.
11. $\tan 2A = \dfrac{\sin A + \sin 3A}{\cos A + \cos 3A}$.
12. $\tan 3A = \dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A}$.

84. Conversely, the sum or difference of any two sines or cosines, or of a sine and cosine, or the difference of their squares, may be transformed at once into a product.

Ex. 1. $\sin(A+B)+\sin(B+C)=2\sin\frac{1}{2}(A+2B+C)\cos\frac{1}{2}(A-C)$.

Ex. 2. $\cos A-\cos(A+2B)=2\sin\frac{1}{2}(2A+2B)\sin\frac{1}{2}(2B)$
$$=2\sin(A+B)\sin B.$$

Ex. 3. $\sin(A-B)+\cos(A+B)=\sin(A-B)+\sin(90^\circ-A-B)$
$$=2\sin\tfrac{1}{2}(90-2B)\cos\tfrac{1}{2}(90-2A)=2\sin(45^\circ-B)\cos(45^\circ-A).$$

Ex. 4. $\sin^2(A-B)-\cos^2 B=-\{\cos^2 B-\sin^2(A-B)\}$
$$=-\cos(B+A-B)\cos(B-A+B)=-\cos A\cos(2B-A).$$

These last transformations are in fact more practically useful than the former, as they enable us to adapt the given quantity in each instance to *logarithmic computation*, for which, as will be seen in the chapter on logarithms, it is necessary to express it in factors. On this account, and because it is very desirable that the Student should be able readily to perform these operations, which are required frequently in Problems, we shall make them the subject of another Exercise.

Ex. 14.

Express in factors the following quantities:

1. $\sin 5A+\sin 3A$; $\cos A-\cos\frac{1}{2}A$.
2. $\sin A+\sin(A-2B)$; $\cos A-\sin(A-B)$.
3. $\sin(A+B)-\sin B$; $2\sin\frac{1}{2}A+2\sin\frac{1}{4}A$.
4. $\operatorname{vers}2A-\operatorname{vers}A$; $\cos(A-B)+\cos(B-C)$.
5. $\sin^2 A-\sin^2(A-2B)$; $\cos^2 A-\cos^2\frac{1}{2}A$.
6. $\cos(2A+B)-\cos(A-2B)$; $\sin A+\sin 2A$.
7. $\operatorname{vers}(A+C)-\operatorname{vers}(B+C)$; $\cos^2 B-\cos^2(A-2B)$.
8. $4\sin^2\frac{1}{2}A-4\sin^2\frac{1}{2}(A-B)$; $\sin A+\cos A$.
9. $\sin^2(A+B)-\cos^2(A+B)$; $\cos^2(A-B)-\sin^2(A+B)$.
10. $\sin(A+B)+\cos(A-B)$; $\sin\frac{1}{2}A-\cos\frac{1}{2}A$.

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85. We may now go on to expand $\sin(A+B+C)$ and $\cos(A+B+C)$, by making use of the formulæ for
$$\sin(A+B) \text{ and } \cos(A+B).$$

Thus
$$\begin{aligned}\sin(A+B+C) &= \sin A\,\cos(B+C)+\cos A\,\sin(B+C)\\ &= \sin A\,(\cos B\,\cos C-\sin B\,\sin C)\\ &\quad+\cos A\,(\sin B\,\cos C+\cos B\,\sin C).\end{aligned}$$

Hence, if A, B, C are the three angles of a triangle, then
$$\sin(A+B+C)=\sin 180^\circ=0;$$
$$\therefore \sin A\,\cos B\,\cos C+\sin B\,\cos A\,\cos C+\sin C\,\cos A\,\cos B = \sin A\,\sin B\,\sin C;$$
or, dividing every term by $\cos A\,\cos B\,\cos C$,
$$\tan A+\tan B+\tan C=\tan A\,\tan B\,\tan C.$$

Like results may be found by expanding $\cos(A+B+C)$, and observing that $\cos(A+B+C)=0$, when A, B, C, are the angles of a triangle. So too, if $A+B+C=90^\circ$, or A, B, C be the *semi*-angles of a triangle, other results may be obtained from the above expansions by observing that $\sin(A+B+C)=1$, $\cos(A+B+C)=0$.

In like manner, if A, B, C, be the angles of a triangle, so that $A+B+C=180^\circ$, or $\tfrac{1}{2}A+\tfrac{1}{2}B+\tfrac{1}{2}C=90^\circ$, then we have
$$\begin{aligned}\cos A+\cos B+\cos C &= 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)+\cos C\\ &= 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}(A-B)+(1-2\sin^2\tfrac{1}{2}C)\\ &= 1+2\sin\tfrac{1}{2}C\left\{\cos\tfrac{1}{2}(A-B)-\sin\tfrac{1}{2}C\right\}\\ &= 1+2\sin\tfrac{1}{2}C\left\{\cos\tfrac{1}{2}(A-B)-\cos\tfrac{1}{2}(A+B)\right\}\\ &= 1+2\sin\tfrac{1}{2}C\left\{2\sin\tfrac{1}{2}A\,\sin\tfrac{1}{2}B\right\}\\ &= 1+4\sin\tfrac{1}{2}A\,\sin\tfrac{1}{2}B\,\sin\tfrac{1}{2}C.\end{aligned}$$

Again, $\tan(A+B+C)$
$$=\frac{\tan A+\tan(B+C)}{1-\tan A\,\tan(B+C)}=\frac{\tan A+\dfrac{\tan B+\tan C}{1-\tan B\,\tan C}}{1-\dfrac{\tan A\,(\tan B+\tan C)}{1-\tan B\,\tan C}}$$
$$=\frac{\tan A+\tan B+\tan C-\tan A\,\tan B\,\tan C}{1-\tan A\,\tan B-\tan A\,\tan C-\tan B\,\tan C}.$$

Hence, if $A+B+C=180°$, or $\tan(A+B+C)=0$, we have the numerator $=0$, or (as before)

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C:$$

and if $A+B+C=90°$, or $\tan(A+B+C)=\infty$, we have the denominator $=0$, or

$$\tan A\tan B+\tan A\tan C+\tan B\tan C=1.$$

Ex. 15.

Prove the following formulæ, where $A+B+C=180°$:

1. $\sin(A+B)\sin(B+C)=\sin A\sin C$.
2. $\cot A\cot B+\cot A\cot C+\cot B\cot C=1$.
3. $\sin(A+B+C)+\sin(A+B-C)+\sin(A+C-B)+\sin(B+C-A)$ $=4\sin A\sin B\sin C$.
4. $\sin A+\sin B+\sin C=4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$.
5. $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$.
6. $\sin^2\frac{1}{2}A+\sin^2\frac{1}{2}B+\sin^2\frac{1}{2}C+2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C=1$.

Prove the following formulæ, where $A+B+C=90°$:

7. $\cot A+\cot B+\cot C=\cot A\cot B\cot C$.
8. $\tan A+\tan B+\tan C=\tan A\tan B\tan C+\sec A\sec B\sec C$.
9. $\sin 2A+\sin 2B+\sin 2C=4\cos A\cos B\cos C$.

For all values of A, B, C, shew that

10. $\cos(A+2B)+\cos(B+2C)+\cos(C+2A)$ $=4\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(B-C)\sin\frac{1}{2}(C-A)$.
11. $\cos(A+B+C)+\cos(B+C-A)+\cos(A+C-B)+\cos(A+B-C)$ $=4\cos A\cos B\cos C$.
12. $\sin(A+B)\sin(B+C)$ $=\sin A\sin C+\sin B\sin(A+B+C)$.

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86. To shew that $\tan^{-1}m\pm\tan^{-1}n=\tan^{-1}\dfrac{m\pm n}{1\mp mn}$.

By the expression $\tan^{-1}m$ is meant *the angle whose tangent is m*; thus if $\tan A=m$, then $A=\tan^{-1}m$. (The origin of this notation is easily seen: for, if we could for a moment conceive the symbol *tan* separated from the angle A, we might write the above equation $A=\dfrac{m}{\tan}$ or by the Theory of Indices, $A=\tan^{-1}.m$.)

Let $\tan A = m$, $\tan B = n$; then $A = \tan^{-1} m$, $B = \tan^{-1} n$;

$$\text{but} \quad \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \, \tan B} = \frac{m \pm n}{1 \mp mn};$$

$$\therefore \tan^{-1} m \pm \tan^{-1} n = A \pm B = \tan^{-1} \frac{m \pm n}{1 \mp mn}.$$

Ex. 1. $2 \tan^{-1} m = \tan^{-1} \dfrac{2m}{1 - m^2}$; $3 \tan^{-1} m = 2 \tan^{-1} m + \tan^{-1} m$

$$= \tan^{-1} \frac{2m}{1 - m^2} + \tan^{-1} m = \tan^{-1} \frac{\dfrac{2m}{1 - m^2} + m}{1 - \dfrac{2m^2}{1 - m^2}} = \tan^{-1} \frac{3m - m^3}{1 - 3m^2}.$$

Ex. 2. $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \tan^{-1} \dfrac{\frac{5}{6}}{\frac{5}{6}} = \tan^{-1} 1$:

hence, if $\tan \alpha = \frac{1}{2}$, $\tan \beta = \frac{1}{3}$, we have $\alpha + \beta = \tan^{-1} 1 = \frac{1}{4}\pi$,

or, in degrees, $A + B = 45°$.

87. We may here observe that the expression $\sin^2 A$, which we have used all along for $(\sin A)^2$, should be more correctly employed to denote $\sin(\sin A)$, that is, the sine of an angle whose value (of course, in circular measure,) is the *number*, $\sin A$. Thus, if $\sin A = \frac{3}{5}$, then $\sin^2 A = \sin(\sin A) = \sin \frac{3}{5}$, that is, $= \sin(\frac{3}{5}\omega) = \sin 34°.4$; and, if $\sin 34°.4$ be found from the Tables, $= \frac{4}{7}$ nearly, then $\sin^3 A = \sin(\sin^2 A) = \sin \frac{4}{7} = \sin 32°.7$, &c. In like manner, $\sin^{-2} m$ is the angle whose sine is $\sin^{-1} m$: thus if $m = \frac{1}{2}$, then $\sin^{-1} m = \frac{1}{6}\pi = .523$ &c; and $\sin^{-2} m$ would be the angle whose sine is .523 &c. But instances of this kind so rarely occur, that no mistakes are likely to arise from using generally the more convenient notation $\sin^2 A$, $\sin^3 A$, &c. for $(\sin A)^2$, $(\sin A)^3$, &c.

88. By means of the formulæ for $\sin A$, $\cos A$, and $\tan A$, in terms of $\sin 2A$ and $\cos 2A$, we may obtain the numerical values of the functions of those angles, which are the halves of 30°, 45°, 60°, &c, and so again, of the halves of these angles.

Ex. 1. $\sin A = \frac{1}{2}\sqrt{(2 - 2\cos 2A)}$ by (76);

$\therefore\ \sin 22^\circ\, 30' = \frac{1}{2}\sqrt{(2 - 2\cos 45^\circ)} = \frac{1}{2}\sqrt{(2 - \sqrt{2})}.$

Ex. 2. $2\cos A = \sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)}$, if A be $< 45^\circ$;

$$\therefore\ 2\cos 11^\circ\, 15' = \sqrt{(1 + \sin 22^\circ\, 30')} - \sqrt{(1 - \sin 22^\circ\, 30')}$$
$$= \sqrt{\{1 + \tfrac{1}{2}\sqrt{(2 - \sqrt{2})}\}} - \sqrt{\{1 - \tfrac{1}{2}\sqrt{(2 - \sqrt{2})}\}}.$$

Ex. 3. $\tan A = \sqrt{\dfrac{1 - \cos 2A}{1 + \cos 2A}}$ by (76);

$$\therefore\ \tan 67^\circ\, 30' = \sqrt{\frac{1 - \cos 135^\circ}{1 + \cos 135^\circ}} = \sqrt{\frac{1 + \dfrac{1}{\sqrt{2}}}{1 - \dfrac{1}{\sqrt{2}}}} = \sqrt{\frac{\sqrt{2} + 1}{\sqrt{2} - 1}} = \sqrt{\frac{(\sqrt{2} + 1)^2}{2 - 1}} = \sqrt{2} + 1.$$

Ex. 16.

Prove the following results:

1. $2\tan^{-1}\frac{1}{2} = \tan^{-1} 1\frac{1}{3}$.

2. $2\tan^{-1}\frac{1}{3} = \cot^{-1} 1\frac{1}{3}$.

3. $\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = \frac{1}{4}\pi$.

4. $\cot^{-1}\frac{3}{4} + \cot^{-1}\frac{1}{7} = \frac{3}{4}\pi$.

5. $\sin^{-1}\sqrt{\frac{3}{5}} = \tan^{-1}\sqrt{\frac{3}{2}}$.

6. $\sin^{-1}\frac{3}{5} + \tan^{-1}\frac{5}{4} = \tan^{-1} 32$.

7. $\sin^{-1}\dfrac{1}{\sqrt{5}} + \cot^{-1} 3 = \frac{1}{4}\pi$.

8. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{1}{4}\pi$.

Obtain the values of the following trigonometrical functions:

9. $\cos 22^\circ\, 30'$.
10. $\cos 67^\circ\, 30'$.
11. $\tan 37^\circ\, 30'$.
12. $\sin 7^\circ\, 30'$.
13. vers 15°.
14. $\cos 11^\circ\, 15'$.
15. $\sin 56^\circ\, 15'$.
16. chd 135°.

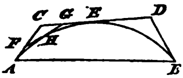
CHAPTER IV.

ON THE CALCULATION OF THE NUMERICAL VALUES OF THE TRIGONOMETRICAL RATIOS.

We shall now proceed to shew how the values of the Trigonometrical Ratios are found for all angles from 0° to 45°, at intervals of 10″: but, in order to this, we must first prove the following propositions.

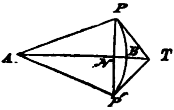
89. *The perimeter of any convex curvilinear figure, contained within another, is less than that of the containing figure.*

Let AEB be a convex curvilinear figure; and if AEB be not shorter than all the lines that can be drawn from A to B, forming a figure exterior to AEB, let $ACEDB$ be taken the *least* of all such lines. Draw a line FHG touching the given convex circumference in H, and cutting CA, CE in F, G: then since CF and CG are together greater than FG, the line $AFGDB$ will be less than the line $ACEDB$, which is absurd. Hence it follows that the perimeter of the convex figure AEB is less than that of any other containing figure.



90. *The circular measure of any angle between* 0° *and* 90° *is greater than its sine and less than its tangent.*

Let BAP, BAP', be equal angles, whose circular measure is θ: with any radius AB describe the circular arc PBP'; draw PNP' perpendicular to AB, and PT, $P'T$, perpendicular to AP, AP'.



Then arc $PBP' >$ line PNP', or $PB > PN$,

$$\text{and, therefore, } \frac{PB}{AP} > \frac{PN}{AP}, \text{ or } \theta > \sin\theta;$$

also (89) arc $PBP' <$ line PTP', or $PB < PT$,

$$\text{and, therefore, } \frac{PB}{AP} < \frac{PT}{AP}, \text{ or } \theta < \tan\theta.$$

91. *As θ is diminished, each of the ratios $\frac{\sin\theta}{\theta}$, $\frac{\tan\theta}{\theta}$, tends more and more to unity, and has unity for its limiting value.*

For θ lies between $\sin\theta$ and $\tan\theta$, and, therefore, $\frac{\theta}{\sin\theta}$ lies between 1 and $\frac{1}{\cos\theta}$; but, as θ is diminished, $\cos\theta$ tends continually to 1, and ultimately becomes 1, when $\theta = 0$; hence $\frac{\theta}{\sin\theta}$, and therefore also $\frac{\sin\theta}{\theta}$, becomes ultimately 1; and hence also $\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta}$ becomes ultimately 1, when $\theta = 0$.

The Student will understand, from the remarks in (48), that the real meaning of such expressions as these is that, as the angle is diminished, its circular measure becomes more and more nearly equal to its sine or its tangent, and, by diminishing the angle, the difference between θ and $\sin\theta$ or $\tan\theta$ may be made as small as we please, without its ever becoming actually zero, because it is impossible that the angle should actually vanish, and its cosine become 1.

Hence also $\cos\theta = (1 - \sin^2\theta)^{\frac{1}{2}} = 1 - \frac{1}{2}\sin^2\theta + \&c = 1 - \frac{1}{2}\theta^2$, when θ is very small.

92. *If θ be the circular measure of an angle between* 0° *and* 90°, *then* $\sin\theta > \theta - \frac{1}{4}\theta^3$.

$\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$; but $\tan\frac{1}{2}\theta > \frac{1}{2}\theta$, or $2\sin\frac{1}{2}\theta > \theta\cos\frac{1}{2}\theta$;

$\therefore \sin\theta > \theta\cos^2\frac{1}{2}\theta > \theta(1 - \sin^2\frac{1}{2}\theta)$, and, *a fortiori*, $> \theta(1 - \frac{1}{4}\theta^2)$.

93. The properties just proved may be at once applied to obtain some interesting results, as in the following Examples.

Ex. 1. A church-tower, seen along a horizontal plain, at a distance of 2 miles, subtends an angle of 1° 5′ 6″: to find approximately its distance.

In the figure of (90), let A be the observer's place, PN the tower: then $\angle PAN = 1^\circ\ 5'\ 6'' = 1^\circ.085$, and its circular measure θ

$$= \frac{1^\circ.085}{180^\circ} \times \pi = \frac{1085}{180000} \times \frac{22}{7} = \frac{341}{18000}:$$

hence $PN = AN \tan\theta = AN \times \theta$ nearly $= \frac{341}{18000} \times 2$ miles $= 200$ feet nearly.

Ex. 2. The Earth's radius (4000 miles) is found, by Astronomical calculations, to subtend at the centre of the Sun an angle of 8″.57116: determine the Sun's distance from the Earth.

In the same figure, let A represent the centre of the Sun, and PN the Earth's radius: then, as before, $PN = AN \times \theta$ nearly,

$$\text{or}\quad AN = PN \div \theta = 4000 \div \frac{8''.57116}{57^\circ.29577} = 4000 \times \frac{206265}{8.57116}$$

$$= 4000 \times 24065.0040 = 96{,}260{,}016 \text{ miles}.$$

In fact the Earth's diameter may be regarded as a small circular arc PP', subtending at the Sun the angle PAP', to which arc we may apply the formula of (29), $a = r\theta$.

Hence the Sun's distance from the Earth, or the radius of the Earth's orbit about the Sun, may be taken as 24,000 times the Earth's radius, or 96 millions of miles.

Ex. 3. Given the Sun's apparent diameter to be $31\frac{1}{2}'$: determine his actual diameter in miles, and compare his bulk with that of the Earth.

By the Sun's *apparent* diameter is meant the angle, or arc (to radius 1), which his actual diameter subtends to an observer upon the Earth, or, rather, to one supposed to be placed at the Earth's centre. Hence if, in the figure of (90), A be taken to represent the Earth's centre, and PNP' the Sun's diameter, then $\angle PAP'$ is his *apparent* diameter; and $PNP' = \text{arc}\,PBP'$ nearly

$$= AP \times \theta = AP \times \frac{31\frac{1}{2}'}{180^\circ} \times \pi = AP \times \frac{31\frac{1}{2}}{180 \times 60} \times \frac{22}{7} = 96{,}000{,}000 \times \frac{11}{1200}$$

$= 880{,}000$ miles, or the Sun's diameter $= 110$ times the Earth's diameter. Hence, since the volumes of similar solids are as the cubes of homologous lines, we have volume of Sun : volume of Earth :: $(\text{diameter})^3$ of Sun : $(\text{diameter})^3$ of Earth :: $(110)^3 : 1$ :: $1{,}331{,}000 : 1$; so that the Sun is more than a million times as large as the Earth.

Ex. 4. The top of a mountain can be just seen at sea at a distance of d miles: determine its height above the level of the sea.

In the figure of (90), let PBP' represent a portion of the Earth's surface, BT the mountain, whose top T can just be seen at the point P, where the line TP touches the circle: then $PB = d = r\theta$, if r be the Earth's radius AP, and θ the circular measure of $\angle PAB$: hence $\theta = \dfrac{d}{r}$, and (91)

$$\cos\theta = 1 - \tfrac{1}{2}\theta^2 = \frac{AP}{AT} = \frac{r}{r+h} = \left(1 + \frac{h}{r}\right)^{-1} = 1 - \frac{h}{r} + \&c;$$

$$\therefore \frac{h}{r} = \tfrac{1}{2}\theta^2 \text{ nearly} = \frac{d^2}{2r^2}, \quad \text{or } h = \frac{d^2}{2r}.$$

If $d = 1$, we have $h = \frac{1}{8000}$ mile $= 8$ inches nearly, so that an object 8 inches high, would be hid by the curvature of the earth at the distance of a mile. This is expressed by saying that the *dip* or *depression* of the horizon is 8 inches for a mile of distance; hence, since $h : h' :: d^2 : d'^2$, or the dip increases as the square of the distance, a hill, 600 feet or 7200 inches high, would be hid at the distance of $d\sqrt{\dfrac{h'}{h}} = \sqrt{\dfrac{7200}{8}} = 30$ miles. Such an approximation is sufficiently exact for ordinary purposes.

Conversely, from the formula $d^2 = 2rh$, may be found the *distance*, when the *height* of the object is given; or the radius of the Earth may be found, if we know by other means the values of d and h in any case.

Ex. 17.

1. Two towers, seen along a horizontal plane, at distances of $2\frac{1}{2}$ and $3\frac{1}{2}$ miles respectively, subtend angles of $1^\circ\, 26'\, 33''$ and $1^\circ\, 36'\, 10''$; compare their heights.

2. The Earth's radius (4000 miles) subtends at the Moon an angle $57'\, 1''.8$: shew that the Moon's distance from the Earth is about 60 times the Earth's radius, or 240,000 miles.

3. The Moon's apparent diameter being 31′ 15″, determine its actual diameter, assuming the result of the last question: and shew that the Earth appears to the Moon about 16 times as large as the Moon appears to us.

4. The diameter of the Earth's orbit subtends at the nearest fixed star an angle of 1″: shew that the star's distance from the Earth must be nearly 20 billions of miles.

5. Shew that the dip of the horizon for three miles of survey is nearly a fathom.

6. At what distance may the Peak of Teneriffe, which is $2\frac{1}{2}$ miles above the level of the sea, be just seen from the deck of a ship?

7. The summit of Dhawalagiri, the highest point of the Himalayas, is 28,000 feet above the level of the sea: at what distance would it first appear in the horizon?

8. If a mountain, 6600 feet high, can be seen at the distance of 100 miles, what must be the Earth's radius?

9. Two points, each 10 feet above the surface of still water, are first visible from each other at a distance of 8 miles: find the Earth's diameter.

10. The hull of a ship is 40 feet above the water, and the masts reach 80 feet above this: at what distance will the hull be hidden from the sight of a person standing on the sea-shore? and at what distance will the whole vessel be lost to view?

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94. *To find the numerical value of* sin 10″.

Let θ be the circular measure of 10″: then, since π is the circular measure of 180°, we have

$$\frac{\theta}{\pi} = \frac{10}{180 \times 60 \times 60} = \frac{1}{64800},$$

$$\text{or } \theta = \frac{\pi}{64800} = \frac{3.14159}{64800} = .00004{,}84813{,}68.$$

Now $\sin 10'' < \theta > \theta - \frac{1}{4}\theta^3$, *a fortiori*, $> \theta - \frac{1}{4}(.00005)$

$$> \theta - .00000\;00000{,}00032,$$

where the quantity to be subtracted does not at all affect the first twelve places of decimals.

Hence to *twelve* places at least the value of $\sin 10''$ is accurately given by the formula

$$\sin 10'' = \text{circ. meas. of } 10'' = \frac{\pi}{64800} = .00004,84813,68;$$

and, therefore, *a fortiori*,

$$\sin 1'' = \text{circ. meas. of } 1'' = \frac{\pi}{648000} = .00000,48481,36.$$

Having thus determined $\sin 10''$, we obtain $\cos 10''$ by the formula $\cos 10'' = \sqrt{(1 - \sin^2 10'')} = .99999,99988,24$.

95. If a be the circular measure of n'', we have

$$a = n \times \text{circular measure of } 1'' = n \sin 1'':$$

and conversely, if a, the circular measure of an angle, be given, we may find n the number of seconds, by the formula $n = \dfrac{a}{\sin 1''}$. This agrees with the Rule in (30); for there the angle was expressed in degrees by multiplying the circular measure by $\dfrac{180}{\pi}$, and, therefore, to bring it into seconds we should have to multiply the circular measure by $\dfrac{180 \times 60 \times 60}{\pi}$, or to divide it by $\dfrac{\pi}{180 \times 60 \times 60} = \dfrac{\pi}{648000} = \sin 1''$, as we have seen above.

96. Knowing now the sine and cosine of $10''$, the sines and cosines of all angles between 0° and 90°, at intervals of $10''$, may be computed as follows.

In the formula, $\sin(A + B) = 2 \sin A \cos B - \sin(A - B)$, put $A = n.10''$, $B = 10''$; then

$$\begin{aligned} \sin(n + 1)10'' &= 2 \sin n.10'' \cos 10'' - \sin(n - 1)10'' \\ &= (2 - k) \sin n.10'' - \sin(n - 1)10'', \end{aligned}$$

if we write $2 - k$, (where k is a very small quantity, $.00000,00023,50$), for $2 \cos 10''$.

$$\text{Hence } \{\sin(n+1)10'' - \sin n.10''\}$$
$$= \{\sin n.10'' - \sin(n-1)10''\} - k\sin n.10'';$$

where by putting, successively, $n = 1, 2, 3$, &c, we get the values of $\sin 20''$, $\sin 30''$, &c: thus

$$(\sin 20'' - \sin 10'') = (\sin 10'' - \sin\ 0'') - k\sin 10'',$$
$$(\sin 30'' - \sin 20'') = (\sin 20'' - \sin 10'') - k\sin 20'',$$
$$\&c. = \&c.$$

The formula is put in the above form, because it is seen that the quantity in brackets on the *second* side is merely a repetition of that which stands upon the *first* side in the preceding line; so that the only additional labour in each line will be to multiply by k the value of $\sin n.10''$ found from the preceding line.

Having thus computed the sines up to 60°, we may proceed to find the rest more simply by mere addition of sines already found, as follows:

$$\sin(60^\circ + A) - \sin(60^\circ - A) = 2\cos 60\ \sin A = \sin A;$$
$$\therefore \sin(60^\circ + A) = \sin(60^\circ - A) + \sin A:$$
$$\text{thus } \sin 60^\circ\, 10'' = \sin 59^\circ\, 59'\, 50'' + \sin 10'',$$
$$\sin 60^\circ\, 20'' = \sin 59^\circ\, 59'\, 40'' + \sin 20'', \&c.$$

And, lastly, having found the sines from 0° to 90°, these give us immediately the cosines from 90° to 0°, since $\cos A = \sin(90^\circ - A)$.

97. The tangents and secants from 0° to 90° may now be found by the formulæ $\tan = \dfrac{\sin}{\cos}$, $\sec = \dfrac{1}{\cos}$; and these will give the cotangents and cosecants from 90° to 0°. But when the tangents have been found up to 45°, the rest may be found more simply by the formula

$$\tan(45^\circ + A) - \tan(45^\circ - A) = 2\tan 2A:$$

thus $\tan 45^\circ\, 10'' = \tan 44^\circ\, 59'\, 50'' + 2\tan 20''$, &c.

Also, since

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{\sin^2\tfrac{1}{2}A + \cos^2\tfrac{1}{2}A}{2\sin\tfrac{1}{2}A\ \cos\tfrac{1}{2}A} = \tfrac{1}{2}(\tan\tfrac{1}{2}A + \cot\tfrac{1}{2}A),$$

we have $\operatorname{cosec} 20'' = \tfrac{1}{2}(\tan 10'' + \cot 10''),$

$$\operatorname{cosec} 40'' = \tfrac{1}{2}(\tan 20'' + \cot 20''), \text{ \&c.}$$

or the cosecants and, therefore, also the secants, may be found for all *even* multiples of 10″, by mere addition from the Tables of tangents and cotangents.

98. As, by the preceding methods, an error, introduced at any point, would be carried on throughout the series of operations, and probably before long begin seriously to affect our results, it is desirable to calculate *directly* some of the Ratios for particular angles, by comparing which with the values for the same Ratios obtained as above, we may check our work at different points, and be certain that we have attained a sufficient degree of accuracy. The Ratios employed for this purpose are the sines and cosines of angles at intervals of 9° from 0° to 90°.

99. *To find the value of* sin 18°.

Let $A = 18°$; then $\sin 36° = \cos 54°$, or $\sin 2A = \cos 3A$;

$$\therefore 2\sin A\ \cos A = 4\cos^3 A - 3\cos A,$$

$$\text{or}\quad 2\sin A = 4\cos^2 A - 3 = 1 - 4\sin^2 A;$$

$$\therefore 4\sin^2 A + 2\sin A - 1 = 0, \quad\text{and}\quad \sin A = \tfrac{1}{4}(\sqrt{5} - 1),$$

where we take the positive sign of the root, because we know that sin 18° is positive.

100. We have now $\sin 18° = \tfrac{1}{4}(\sqrt{5} - 1) = \cos 72°$, whence

$$\cos 18° = \sqrt{(1 - \sin^2 18°)} = \sqrt{\{1 - \tfrac{1}{16}(6 - 2\sqrt{5})\}} = \tfrac{1}{4}\sqrt{(10 + 2\sqrt{5})} = \sin 72°$$

$$\cos 36° = 1 - 2\sin^2 18° = 1 - \tfrac{1}{4}(3 - \sqrt{5}) = \tfrac{1}{4}(\sqrt{5} + 1) = \sin 54°$$

$$\sin 36° = \sqrt{(1 - \cos^2 36°)} = \sqrt{\{1 - \tfrac{1}{16}(6 + 2\sqrt{5})\}} = \tfrac{1}{4}\sqrt{(10 - 2\sqrt{5})} = \cos 54°$$

Also putting 9° and 27° for A in the formulæ

$$\sin A = \tfrac{1}{2}\{\sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}\},$$
$$\cos A = \tfrac{1}{2}\{\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}\},$$

we get

$$\sin 9^\circ = \tfrac{1}{4}\{\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}\} = \cos 81^\circ,$$
$$\cos 9^\circ = \tfrac{1}{4}\{\sqrt{(3+\sqrt{5})} + \sqrt{(5-\sqrt{5})}\} = \sin 81^\circ,$$
$$\sin 27^\circ = \tfrac{1}{4}\{\sqrt{(5+\sqrt{5})} - \sqrt{(3-\sqrt{5})}\} = \cos 63^\circ,$$
$$\cos 27^\circ = \tfrac{1}{4}\{\sqrt{(5+\sqrt{5})} + \sqrt{(3-\sqrt{5})}\} = \sin 63^\circ.$$

And thus, remembering that $\sin 45^\circ = \tfrac{1}{2}\sqrt{2} = \cos 45^\circ$, we have the values of the sine and cosine for all angles at intervals of 9° from 0° to 90°, which, by extracting the roots indicated, may be expressed as simple decimals.

101. The following are called *Formulæ of Verification.*

$$\sin(36^\circ+A) - \sin(36^\circ-A) = 2\cos 36^\circ \sin A = \tfrac{1}{2}(\sqrt{5}+1)\sin A,$$
$$\sin(72^\circ+A) - \sin(72^\circ-A) = 2\cos 72^\circ \sin A = \tfrac{1}{2}(\sqrt{5}-1)\sin A;$$
$$\therefore \sin A = \sin(36^\circ+A) - \sin(36^\circ-A) - \sin(72^\circ+A) + \sin(72^\circ-A),$$

a relation between the sines of five angles, which, if found to be satisfied by our tabulated results, will prove them to be sufficiently accurate.

In like manner

$$\cos A = \cos(36^\circ+A) + \cos(36^\circ-A) - \cos(72^\circ+A) - \cos(72^\circ-A).$$

Since $\sin(36^\circ+A) = \cos(90^\circ-36^\circ-A) = \cos(54^\circ-A)$, &c, these two formulæ may also be written

$$\sin A = \cos(54^\circ-A) - \cos(54^\circ+A) - \cos(18^\circ-A) + \cos(18^\circ+A),$$
$$\cos A = \sin(54^\circ-A) + \sin(54^\circ+A) - \sin(18^\circ-A) - \sin(18^\circ+A).$$

102. The Tables do not go beyond 45°: the values of the sines, tangents, and secants for angles above 45° being the same as those of the cosines, cotangents, and cosecants of their complements. Thus the page which contains the values of the Ratios from 19° to 20°, is marked *at the top* with 19°, and on the left hand is

a *descending* column of minutes; while at the bottom it is marked with 70°, and on the right is an *ascending* column of minutes; and the columns, which are marked sin, cos, tan, &c. at the top, are marked cos, sin, cot, &c. at the bottom: so that (for instance) the same numerical value would be read from the top as that of sin 19° 54′, and from the bottom as that of cos 70° 6′.

Since the values of the sine and cosine, or of the tangents of angles less than 45°, are always less than 1, and therefore will be decimal fractions, sometimes with two or three cyphers after the decimal point, they are printed in some Tables with the decimal point moved four places to the right, (as if they were each multiplied by 10,000), while in others the decimal point is omitted altogether. No mistake, however, can arise in taking out the true value, if the Student is only made aware of this fact.

All the above values are called the *Natural* sines, cosines, &c., to distinguish them from what are commonly called the *Logarithmic* sines, (or shortly, *log-sines*,) &c. which, however, would be more properly called the *Logarithms of the* (Natural) *sines*, &c. These latter are by far the most frequently required in practice: more will be said about them in the chapter on Logarithms.

103. All Tables, however, are not given so accurately as to 10″; but, in such cases, the values of the sine, &c. for any angle not exactly given in the Tables, may be found with sufficient accuracy for all ordinary purposes by a simple Proportion: and, conversely, if we have given the value of a Ratio, which we cannot find exactly in the Tables, we may, in like manner, obtain the proper angle.

Thus, let the interval at which the angles are given in any set of Tables be a'', and let D be the difference between any (the same) functions of two successive angles, A and $A + a''$, D being obtained by subtracting the function of the lesser angle A from that of $A + a''$; and let d be the difference similarly obtained between the same functions of A and $A + n''$: then, assuming that $D : d :: a : n$, which (as will be presently shewn) is in most cases very nearly true, we get $d = \frac{n}{a} \times D$, $n = \frac{d}{D} \times a$.

If $a = 60$, as in Hutton's Tables, where the angles are given at intervals of 1′, these become

$$d = \frac{n}{60} \times D, \quad n = \frac{d}{D} \times 60.$$

Ex. 1. To find the value of sin 37° 23′ 42″ from Hutton's Tables.

$$\text{Here} \quad \left.\begin{array}{l}\sin 37^\circ\, 24' = .6073758 \\ \sin 37\ \ 23' = \underline{.6071447}\end{array}\right\} \text{ as taken out from the Tables;}$$

D = Diff. for 1′ or 60″ = + 2311 : $\therefore d$ = diff. for 42″ = $\frac{42}{60} \times 2311 = 1618$;

$$\therefore \sin 37^\circ\, 23'\, 42'' = .6071447 + .0001618 = .6073065.$$

Ex. 2. To find cot 55° 27′ 48″.

$$\begin{array}{rl}\text{Here } \cot 55^\circ\, 28' & = .6881379 \\ \cot 55^\circ\, 27' & = \underline{.6885666}\end{array}$$

D = Diff. for 60″ = − 4287; $\therefore d$ = diff. for 48″ = $\frac{48}{60} \times -4287 = -3430$;

$$\therefore \cot 55^\circ\, 27'\, 48'' = .6885666 - .0003430 = .6882236.$$

N.B. Notice that the Diff. will always be negative for the cosine, cotangent, and cosecant, which all *decrease* as the angle increases from 0° to 90°.

Ex. 3. Given $\sec A' = 9.7654237$: to find A' or $\sec^{-1} 9.7654237$ accurately to 1″, from Tables calculated at intervals of 1′.

Here, by the Tables, it would be seen that A' lies between 84° 7′ and 84° 8′, or $A' = 84^\circ\, 7'\, n'' = A + n''$, suppose.

$$\begin{array}{lrl}\text{Now} & \sec(A + 1') & = 9.7834124 \\ & \sec A & = \underline{9.7557944}\end{array} \qquad \begin{array}{rl}\text{and } \sec(A + n'') & = 9.7654237 \\ \sec A & = \underline{9.7557944}\end{array}$$

$\therefore$ Diff. for 1′ or 60″ = + 276180; diff. for n'' = + 96293:

hence $n = \frac{96293}{276180} \times 60 = 20.9 = 21$ nearly, and $A' = 84^\circ\, 7'\, 21''$.

Ex. 4. Given $\cos A' = .8241657$: to find A' or $\cos^{-1}.8241657$ from the same Tables.

Here, by the Tables, A' lies between $34^\circ\ 29'$ and $34^\circ\ 30'$, or $A' = 34^\circ\ 29'\ n'' = A + n''$, suppose.

$$\begin{array}{lrlr} \text{Now} & \cos(A+1') = .8241262 & \text{and} & \cos(A+n'') = .8241657 \\ & \cos A = \underline{.8242909} & & \cos A = \underline{.8242909} \\ \therefore \text{Diff. for } 1' \text{ or } 60'' = & -1647; & \text{diff. for } n'' = & -1252: \end{array}$$

hence $n = \frac{1252}{1647} \times 60 = 45.6 = 46$ nearly, and $A' = 34^\circ\ 29'\ 46''$.

Ex. 18.

1. Given $\sin 24^\circ\ 37' = .4165453$, $\sin 24^\circ\ 38' = .4168097$;
   find $\sin 24^\circ\ 37'\ 15''$, and $\sin^{-1}.4166287$.
2. Given $\cos 68^\circ\ 12' = .3713678$, $\cos 68^\circ\ 13' = .3710977$;
   find $\cos 68^\circ\ 12'\ 24''$, and $\cos^{-1}.3711999$.
3. Given $\tan 45^\circ\ 1' = 1.0005819$;
   find $\tan 45^\circ\ 0'\ 35''$, and $\tan^{-1} 1.0002345$.
4. Given $\cot 30^\circ\ 1' = 1.7308878$;
   find $\cot 30^\circ\ 0'\ 17''$, and $\cot^{-1} 1.7312123$.
5. Given $\sec 59^\circ\ 59' = 1.9989929$;
   find $\sec 59^\circ\ 59'\ 25''$, and $\sec^{-1} 1.9990678$.
6. Given $\operatorname{cosec} 60^\circ\ 1' = 1.1545067$;
   find $\operatorname{cosec} 60^\circ\ 0'\ 37''$, and $\operatorname{cosec}^{-1} 1.1546025$.

Prove the following results:

7. $\operatorname{vers}(A-B) \operatorname{suvers}(A+B) = (\sin A - \sin B)^2$.
8. $\tan(45^\circ + \frac{1}{2}A) + \cot(45^\circ + \frac{1}{2}A) = 2\sec A$.
9. $\tan(30^\circ + A)\tan(30^\circ - A) = \dfrac{2\cos 2A - 1}{2\cos 2A + 1}$.
10. $\tan^4 A = \dfrac{\sin^2 2A - 4\sin^2 A}{\sin^2 2A + 4\sin^2 A - 4}$.
11. If $1 = \dfrac{\sin^2 A}{\sin^2 B} + \cos^2 A \cos^2 C$, shew that $\sin C = \dfrac{\tan A}{\tan B}$.
12. Determine B from the equation
    $\sin A + \sin(2B + A) - \sin(2B - A) = \sin(B + A) - \sin(B - A)$.

If A, B, C, are in A.P., prove the following formulæ:

13. $\sin A - \sin C = 2\sin(A - B)\cos B$.
14. $(\sin A - \sin C)\sin B = (\cos C - \cos A)\cos B$.

104. We shall here prove the truth of the principle assumed in (103), namely, that for a very small increment of the angle, the increment of its sine, cosine, &c. is, in general, proportional to the increment of the angle.

$$\begin{aligned}\sin(\theta+\delta) - \sin\theta &= \sin\theta\cos\delta + \cos\theta\sin\delta - \sin\theta\\ &= \cos\theta\sin\delta - \sin\theta(1-\cos\delta)\\ &= \cos\theta\sin\delta\,(1-\tan\theta\tan\tfrac{1}{2}\delta) \text{ by (74).} \qquad (1)\end{aligned}$$

$$\begin{aligned}\cos(\theta+\delta) - \cos\theta &= \cos\theta\cos\delta - \sin\theta\sin\delta - \cos\theta\\ &= -\sin\theta\sin\delta - \cos\theta\,(1+\cos\delta)\\ &= -\sin\theta\sin\delta\,(1+\cot\theta\cot\tfrac{1}{2}\delta). \qquad (2)\end{aligned}$$

$$\begin{aligned}\tan(\theta+\delta) - \tan\theta &= \frac{\sin(\theta+\delta)}{\cos(\theta+\delta)} - \frac{\sin\theta}{\cos\theta} = \frac{\sin(\theta+\delta)\cos\theta - \cos(\theta+\delta)\sin\theta}{\cos\theta\cos(\theta+\delta)}\\ &= \frac{\sin(\theta+\delta-\theta)}{\cos\theta\cos(\theta+\delta)} = \frac{\sin\delta}{\cos^2\theta\cos\delta\,(1-\tan\theta\tan\delta)}\\ &= \frac{\sec^2\theta\tan\delta}{1-\tan\theta\tan\delta}. \qquad (3)\end{aligned}$$

Now, if δ be very small, $\sin\delta$ and $\tan\delta$ are by (91) very nearly equal to δ, and both, therefore, very small; hence, unless $\tan\theta$ be large in (1) and (3), or $\cot\theta$ in (2), we may write the above

$$\sin(\theta+\delta) - \sin\theta = \delta\cos\theta, \quad \cos(\theta+\delta) - \cos\theta = -\delta\sin\theta,$$
$$\tan(\theta+\delta) - \tan\theta = \delta\sec^2\theta;$$

where we see that, for any given value of θ, the increment of the sine and tangent are proportional to that of the angle, supposed very small, provided, however, that $\tan\theta$ be not very great, or θ near 90°; and similarly for the cosine, provided that $\cot\theta$ be not very great, or θ near 0° or 180°.

Similarly it may be shewn that

$$\cot(\theta+\delta) - \cot\theta = \frac{-\sin\delta\,\mathrm{cosec}^2\theta}{1+\cot\theta\tan\delta},$$

$$\sec(\theta+\delta) - \sec\theta = \frac{\tan\delta\sec\theta\,(\tan\theta+\tan\tfrac{1}{2}\delta)}{1-\tan\theta\tan\delta},$$

$$\mathrm{cosec}(\theta+\delta) - \mathrm{cosec}\,\theta = \frac{-\tan\delta\,\mathrm{cosec}\,\theta\,(\cot\theta-\tan\tfrac{1}{2}\delta)}{1+\cot\theta\tan\tfrac{1}{2}\delta}.$$

So that here also we may write

$$\cot(\theta+\delta) - \cot\theta = -\delta\,\mathrm{cosec}^2\theta, \quad \sec(\theta+\delta) - \sec\theta = \delta\sec\theta\tan\theta,$$
$$\mathrm{cosec}(\theta+\delta) - \mathrm{cosec}\,\theta = -\delta\,\mathrm{cosec}\,\theta\cot\theta,$$

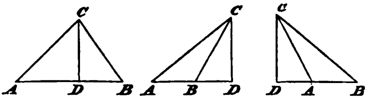
unless θ be near 0° or 180° for the cotangent and cosecant, or near 90°, for the secant.

CHAPTER V.

ON THE TRIGONOMETRICAL PROPERTIES OF TRIANGLES, QUADRILATERALS, AND POLYGONS.

105. *To shew that in any triangle the sides are proportional to the sines of the opposite angles.*

In future we shall use the letters a, b, c, to denote the sides BC, AC, AB, opposite to the angles A, B, C, respectively, of any triangle ABC.



Let ABC be any triangle, and from C draw CD perpendicular on AB, or on AB produced: then

$$CD = AC \sin A = BC \sin B, \text{ or } \frac{\sin A}{\sin B} = \frac{BC}{AC} = \frac{a}{b}.$$

Similarly, by drawing a perpendicular from B on AC, we may shew that $\dfrac{\sin A}{\sin C} = \dfrac{BC}{AB} = \dfrac{a}{c}$.

Hence we have $a : b : c :: \sin A : \sin B : \sin C$;

or, as it may be also written, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

106. *To express the cosine of an angle of a triangl in terms of its sides.*

Let ABC be any triangle, and from C, as before draw CD perpendicular on AB, or on AB produced.

Then in figs. 1 and 2,

$$BC^2 = AC^2 + AB^2 - 2AB.AD, \quad (Euc. \text{ II } 13)$$

in fig. 3,

$$BC^2 = AC^2 + AB^2 + 2AB.AD: \quad (Euc. \text{ II } 12)$$

but $AD = AC \cos CAD = AC \cos A$ in figs. 1 and 2,

$= -AC \cos A$ in fig. 3;

$\therefore$ in each case, $BC^2 = AC^2 + AB^2 - 2AB.AC \cos A$,

or $a^2 = b^2 + c^2 - 2bc \cos A$, whence $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Of course, it was not strictly necessary to have *proved* (as we have done) the truth of the above in *all* cases; since the result with the first figure would have served for all the rest by the principle of (6).

In like manner we should get

$$b^2 = a^2 + c^2 - 2ac \cos B, \quad c^2 = a^2 + b^2 - 2ab \cos C,$$

$$\text{and } \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

107. To shew that, if $s = \frac{1}{2}(a + b + c)$, then

$$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}, \quad \sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}};$$

and thence to deduce expressions for $\tan\frac{1}{2}A$ and $\sin A$.

We have

$$2\cos^2 \tfrac{1}{2}A = 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b + c + a)(b + c - a)}{2bc} = \frac{2s(2s - 2a)}{2bc};$$

$$\text{and } 2\sin^2 \tfrac{1}{2}A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a - b + c)(a + b - c)}{2bc} = \frac{(2s - 2b)(2s - 2c)}{2bc};$$

$$\therefore \cos^2 \tfrac{1}{2}A = \frac{s(s-a)}{bc}, \quad \sin^2 \tfrac{1}{2}A = \frac{(s-b)(s-c)}{bc},$$

$$\text{and } \cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}, \quad \sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

where the positive roots are taken, because A, being an angle of a triangle, must be < 180°, and $\frac{1}{2}A$ < 90°.

Hence $\tan\frac{1}{2}A = \dfrac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$;

and $\sin A = 2\sin\frac{1}{2}A\cos\frac{1}{2}A = \dfrac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}}$.

From the last result it appears that $\dfrac{\sin A}{a}$ is a symmetrical expression in terms of a, b, c; so that we find

$$\frac{\sin A}{a} = \frac{2}{abc}\sqrt{\{s(s-a)(s-b)(s-c)\}} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

as, of course, we should expect from (105).

In like manner we should get

$$\cos\tfrac{1}{2}B = \sqrt{\frac{s(s-b)}{ac}}, \quad \sin\tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}},$$

$$\cos\tfrac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}, \quad \sin\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}};$$

with corresponding expressions for $\tan\frac{1}{2}B$ and $\sin B$, $\tan\frac{1}{2}C$ and $\sin C$, which may be written down at once, by *Symmetry*, from those for $\tan\frac{1}{2}A$ and $\sin A$.

108. The formulæ proved in this chapter are true for all triangles whatsoever; but they become simplified when applied to a right-angled triangle, so as to agree with our former expressions.

Thus, if $C = 90°$, then $\sin C = 1$, and $\dfrac{\sin A}{a} = \dfrac{1}{c} = \dfrac{\sin B}{b}$,

$$\text{or } a = c\sin A,\ b = c\sin B:$$

so $\cos C = 0 = \dfrac{a^2+b^2-c^2}{2ab}$; $\therefore a^2+b^2=c^2$, (as in *Euc.* I. 47);

and $\therefore \cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{2b^2}{2bc} = \dfrac{b}{c}$, or $b = c\cos A$,

and, similarly, $a = c\cos B$.

109. *To express the area of a triangle in terms of the sides and angles.*

Referring to the figures in (105), since the area of every triangle = $\frac{1}{2}$ rectangle of same base and height = $\frac{1}{2}$ base × height, we shall have

$$\text{area of triangle } ABC = \tfrac{1}{2}AB.CD = \tfrac{1}{2}AB.AC\sin A$$
$$= \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}ab\sin C.$$

Hence also the area = by (107)

$$\frac{bc}{2}\times\frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}} = \sqrt{\{s(s-a)(s-b)(s-c)\}}$$
$$= \tfrac{1}{4}\sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)};$$

$$\text{or } = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}\,\frac{a\sin C}{\sin A}\,\frac{a\sin B}{\sin A}\sin A = \tfrac{1}{2}a^2\,\frac{\sin B\sin C}{\sin(B+C)}.$$

110. Since $\sin A$, $\sin B$, $\sin C$, are respectively proportional to a, b, c, we may substitute the former quantities for the latter (or *vice versa*) in cases where they occur, involved *homogeneously*, either on opposite sides of an equation, or in the num^r^ and den^r^ of a fraction. This step will often be found of great use in the solution of Problems, and will be best illustrated by the following Examples. (See *Alg.*, Part I 85–88, and Part II 26–28.)

Ex. 1. Since $\sin C = \sin(A+B) = \sin A\cos B + \sin B\cos A$, we have, at once, $c = a\cos B + b\cos A$:

$$\text{for } \frac{\sin C}{\sin A} = \cos B + \frac{\sin B}{\sin A}\cos A, \text{ or } \frac{c}{a} = \cos B + \frac{b}{a}\cos A, \text{ \&c.}$$

$$\text{Ex. 2. } \frac{a^2}{b+c} = \frac{a\sin A}{\sin B + \sin C}, \text{ or } = (b+c)\frac{a^2}{(b+c)^2} = \frac{(b+c)\sin^2 A}{(\sin B + \sin C)^2}.$$

We may best *prove* the truth of such transformations by putting

$$\frac{a}{\sin A} = x = \frac{b}{\sin B} = \frac{c}{\sin C}, \text{ or } a = x\sin A,\ b = x\sin B,\ c = x\sin C$$

$$\therefore \frac{a^2}{b+c} = \frac{ax\sin A}{x\sin B + x\sin C} = \frac{a\sin A}{\sin B + \sin C}, \text{ \&c.}$$

Ex. 3. Area = $\frac{1}{2}bc\sin A$

$$= \tfrac{1}{2}(b^2+c^2)\frac{bc\sin A}{b^2+c^2} = \tfrac{1}{2}(b^2+c^2)\frac{\sin A\sin B\sin C}{\sin^2 B + \sin^2 C}.$$

Ex. 19.

In any triangle, *right-angled* at C, prove the following formulæ:

1. $\cos(A - B) = \dfrac{2ab}{c^2}$. 2. $\cos 2A = \dfrac{b^2 - a^2}{b^2 + a^2}$.

3. $\tan 2B = \dfrac{2ab}{a^2 - b^2}$. 4. $\tan \frac{1}{2}(A - B) = \dfrac{a - b}{a + b}$.

5. $\cos(2A - B) = \dfrac{a}{c^3}(3c^2 - 4a^2)$. 6. $\tan \frac{1}{2}A = \sqrt{\dfrac{c - b}{c + b}}$.

In the same triangle, obtain the following expressions for the area:

7. $\frac{1}{4}(a + b + c)(a + b - c)$. 8. $\frac{1}{4}c^2 \sin 2A$.

In any triangle prove the truth of the following formulæ:

9. $\sin(A-B) = \dfrac{a^2-b^2}{c^2}\sin(A+B)$. 10. $\dfrac{\operatorname{vers} A}{\operatorname{vers} B} = \dfrac{a(a+c-b)}{b(b+c-a)}$.

11. $\dfrac{\tan B}{\tan C} = \dfrac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$. 12. $\sin\frac{1}{2}(A - B) = \dfrac{a - b}{c}\cos\frac{1}{2}C$.

13. $\tan\frac{1}{2}A \tan\frac{1}{2}B = \dfrac{a + b - c}{a + b + c}$. 14. $\dfrac{\tan\frac{1}{2}A + \tan\frac{1}{2}B}{\tan\frac{1}{2}A - \tan\frac{1}{2}B} = \dfrac{c}{a - b}$.

In any triangle, obtain the following expressions for the *area:*

15. $\frac{1}{2}(a^2 - b^2)\dfrac{\sin A \sin B}{\sin(A - B)}$. 16. $\dfrac{2abc}{a + b + c}\cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C$.

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111. There are *six* parts of a triangle, three sides and three angles; and there are *three* necessary relations, which exist between these parts from the very nature of the case, or they could not possibly belong to a triangle at all. These relations are expressed algebraically by the following equations, (to which we are led without considering the properties of any *particular* triangle, but merely such as are common to all triangles whatsoever,) namely,

$$A + B + C = 180°, \text{ and } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

All other results we have obtained are in fact only modifications of these.

Thus we may derive the expression for $\cos A$ as follows:

By (110 Ex. 1), we have found, merely using the above formulæ, that
$$c = a \cos B + b \cos A:$$

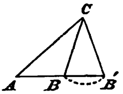
hence $c - b \cos A = a \cos B$, or $c^2 - 2bc \cos A + b^2 \cos^2 A = a^2 \cos^2 B$, whence $c^2 - 2bc \cos A + b^2 - a^2 = b^2 \sin^2 A - a^2 \sin^2 B = 0$ by (105); and thus we get $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, as before.

112. Generally then it will be sufficient, for the complete determination of all the parts of a triangle, if we have given three of the six parts, or three independent equations between them, since then we shall have six equations for determining the six parts. But there are two exceptions to this statement.

(1) If the *three* angles alone are given, we have in reality only *five* equations given us; for, when *two* angles are given, the third is at once known from $A + B + C = 180°$, and, therefore, it is useless to give it as a third datum. In this case the ratios of the sides will be determinate, and we may find them from the equations $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$, $\dfrac{a}{c} = \dfrac{\sin A}{\sin C}$; so that the triangles, corresponding to our data, will be all *similar*, and be determined in *species*, though not in actual magnitude. Consequently, *one* side, at least, must always be given, in order to determine a triangle.

(2) If a, b, A are given, that is, *two sides and the angle opposite to one of them*, then, if $a < b$, or the side opposite to the given angle be less than the other given side, it will be shewn that there are *two* triangles, which have the same given data, and the problem, therefore, is *ambiguous*, or admits of a double solution. This, however, can only happen when $A < 90°$; for, since $b > a$, therefore also $B > A$, and, consequently, A must be $< 90°$, or else we should have two angles of a triangle together *greater than* two right angles.

13



Take $AC = b$, and $\angle CAB (< 90^\circ) = A$; with centre C and radius $= a$, describe a circular arc: then, if $a < b$, it is plain that the circle will cut AB in two points B, B', on the same side of AC as that on which is formed the angle A, and that each of the triangles CAB, CAB', will have the three data, a, b, A. It may be observed that

$$\angle CBA = 180^\circ - \angle CBB' = 180^\circ - \angle CB'B;$$

that is, the angle B in one triangle will be the supplement of the angle B' in the other.

If, however, b be such that the points B, B', coincide, then there will be but one triangle, and CB will be perpendicular to AB, and $\therefore CB = AC \sin A$, or $a = b \sin A$: so that, if we see that the given data satisfy this equation, we may infer at once that the case is not ambiguous, although a be less than b, and that the triangle is, in fact, right-angled.

113. If in (112) the radius (a) be *less* than the perpendicular ($b \sin A$) from C upon AB, then it is plain that the circle will never cut AB, and no triangle at all can be formed. In such a case then the solution is *impossible;* and also if, in any case, two angles are given, whose sum is greater than two right angles, or three sides, any two of which are not together greater than the third.

In all other cases the triangle may be solved, subject to the exceptions of (112). The examples here given will not require the aid of logarithms: but the Student should read the chapter on Logarithms, before proceeding to the next chapter upon the general solution *of triangles.*

Ex. 20.

1. One angle of a triangle is 30°, and the sides including it are $2\frac{2}{3}$ and $3\frac{3}{4}$ yards, respectively: determine its area.

2. Find the area of an equilateral triangle, one of whose sides is 60 yards.

3. Find the area of an isosceles triangle, whose base is 50 feet, and each of its equal sides 30 feet.

4. How many trees can be planted in a triangular piece of ground, the lengths of whose sides are 130, 120, and 50 yards, if two yards in length and breadth be allowed for each tree?

5. Find the area of a triangular field, whose sides are 216 yards, 270 yards, and 162 yards.

6. What length (AE) must be taken along the diagonal of a square field ($ABCD$), whose side is 100 yards, so that the triangle AEB may be exactly a fifth part of the whole square?

7. Find the area of a parallelogram, whose adjacent sides are 28 and 30 feet, and the included angle 75°.

8. The sides of a triangular field measure 189, 169, and 42 yds: find its value at £150 an acre.

9. The hypothenuse of a right-angled triangle is 5.25 feet, and one of its angles is 30°: find the remaining parts, and the area of the triangle.

10. The angles of a triangle are in A.P., the least being 30°, and the opposite side is 100 yards: find the area.

11. Given $a = 10$, $c = 20$, $A = 30°$; solve completely the triangle.

12. The sides of a triangle are as 1, $1\frac{1}{3}$, $1\frac{2}{3}$; find the greatest angle, and the sines of the other two.

13. If, in any triangle, $a^2 = b^2 + bc + c^2$, shew that $A = 120°$.

14. Apply each of the formulæ for $\sin\frac{1}{2}C$, $\cos\frac{1}{2}C$, and $\tan\frac{1}{2}C$, to shew that, when $C = 90°$, $a^2 + b^2 = c^2$.

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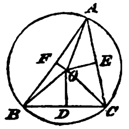
114. *To find the radius of the circumscribed circle of a triangle, in terms of its sides and angles.*



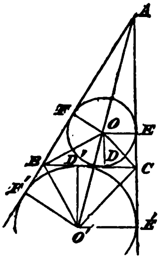
Let ABC be any triangle, $OA = OB = OC = R$, the radius of its *circumscribed* circle, whose centre O is found (*Euc.* IV 5) by *bisecting the sides* of the triangle in D, E, F, and *drawing* the perpendiculars DO, EO, FO.

Then $BD = BO \sin BOD$, or $\frac{1}{2}a = R \sin A$, since
$$\angle BOD = \tfrac{1}{2}\angle BOC = \angle BAC = A\,;$$
hence we have $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$

or (in sides only) $= \dfrac{abc}{4\sqrt{\{s(s-a)(s-b)(s-c)\}}} = \dfrac{abc}{4S}$,

if S represent the *Surface*, or area, of the triangle ABC.

115. *To find the radius of the inscribed and escribed circles of a triangle, in terms of its sides and angles.*



Let ABC be any triangle, and let
$$OD = OE = OF = r$$
be the radius of its *inscribed* circle, whose centre O is found (*Euc.* IV 4) by *bisecting the angles* of the triangle by the lines AO, BO, CO, meeting in O.

Then we shall have the area ABC = area OBC + area OAC + area OAB,

or $\frac{1}{2}bc \sin A = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$;

$\therefore\ bc \sin A = (a + b + c)\,r = 2sr$, and $r = \dfrac{bc}{2s}\sin A$,

or (in sides only) $= \dfrac{bc}{2s} \times \dfrac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}} = \dfrac{S}{s}$.

Similarly, if the exterior angles at B and C be bisected by BO', CO', then O' will be the centre of a circle, which will touch BC and the other two sides AB, AC, *produced*, and which is called an *escribed* circle of the triangle. Now if the radius of this circle $O'D' = O'E' = O'F' = r_1$, we shall have

area ABC = area $O'AC$ + area $O'AB$ − area $O'BC$,

or $\frac{1}{2}bc \sin A = \frac{1}{2}br_1 + \frac{1}{2}cr_1 - \frac{1}{2}ar_1$;

hence $bc \sin A = (b + c - a)\,r_1 = 2(s - a)\,r_1$,

$$\text{or } r_1 = \frac{bc}{2(s-a)}\sin A = \frac{S}{s-a}.$$

In like manner, if r_2, r_3, be the radii of the escribed circles, touching the sides b and c respectively, then

$$r_2 = \frac{S}{s-b}, \quad r_3 = \frac{S}{s-c}.$$

N.B. Notice, for the sake of problems, that, since

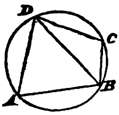
$$AE = AF, \quad BD = BF, \quad CD = CE,$$

$$\therefore AE + BF + CD = \tfrac{1}{2}(\text{perimeter}) = s;$$

$$\text{and } AE = s - (BF + CD) = s - (BD + CD) = s - a:$$

$$\text{hence } AE = AF = s - a, \quad BD = BF = s - b, \quad CD = CE = s - c.$$

116. *To find the area of a quadrilateral, whose opposite angles are supplementary, or, in other words, of a quadrilateral which may be inscribed in a circle.*



Let $ABCD$ be a quadrilateral, such that the angles A and C, and, therefore, also B and D, are supplementary, and which may, consequently, (*Euc.* III 22) be inscribed in a circle.

Let $AB = a$, $BC = b$, $CD = c$, $DA = d$, and join BD:

$$\text{then area } ABCD = \text{area } ABD + \text{area } CBD$$

$$= \tfrac{1}{2}ad \sin A + \tfrac{1}{2}bc \sin C = \tfrac{1}{2}(ad + bc)\sin A.$$

$$\text{Now we have } 2ad \cos A = a^2 + d^2 - BD^2$$

$$-2bc \cos A = b^2 + c^2 - BD^2,$$

$$\text{since } \cos C = \cos(180^\circ - A) = -\cos A:$$

$$\text{and hence } 2(ad + bc)\cos A = a^2 + d^2 - b^2 - c^2,$$

$$\text{or } \cos A = \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)};$$

$$\therefore 1 - \cos A = 1 - \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)} = \frac{(b + c)^2 - (a - d)^2}{2(ad + bc)},$$

$$1 + \cos A = 1 + \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)} = \frac{(a + d)^2 - (b - c)^2}{2(ad + bc)};$$

and $\sin^2 A = 1 - \cos^2 A = \frac{(b+c)^2-(a-d)^2}{2(ad+bc)} \times \frac{(a+d)^2-(b-c}{2(ad+bc)}$

$$= \frac{(b+c+a-d)\,(b+c+d-a)\,(a+d+b-c)\,(a+d+c-b}{4\,(ad+bc)^2}$$

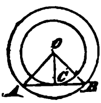
$$= \frac{2\,(s-d)\,2\,(s-a)\,2\,(s-c)\,2\,(s-b)}{4\,(ad+bc)^2},$$

(if $2s = a + b + c + d$, and $\therefore 2s - 2a = b + c + d - a$

hence $\sin A = \frac{2}{ad+bc}\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$

and area $ABCD = \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$.

117. *To find the area of any regular polygon, an the radii of its inscribed and circumscribed circles.*



Let AB (a) be the side of a regular polygon n sides, O the common centre of its inscrib and circumscribed circles. Draw OC pe pendicular on AB: then, since the sum of the angles, subtended at O by the sides the polygon, is 360° or 2π, we have $\angle AOB = \frac{2\pi}{n}$.

Hence R = radius of circle circumscribed

$$= OA = AC \operatorname{cosec} AOC = \tfrac{1}{2}a \operatorname{cosec} \frac{\pi}{n};$$

and r = radius of circle inscribed

$$= OC = AC \cot AOC \quad = \tfrac{1}{2}a \cot \frac{\pi}{n}:$$

also area of polygon

$$= n \times \text{area} OAB = n \times \tfrac{1}{2}AB.OC = \tfrac{1}{4}na^2 \cot \frac{\pi}{n}.$$

118. Conversely, if R or r be given, then the side the regular polygon of n sides, which can be describ *in* a circle, radius R, or *about* a circle, radius r, will from (117) $2R \sin \frac{\pi}{n}$ or $2r \tan \frac{\pi}{n}$, respectively.

So also area of polygon, in terms of R,

$$= n \times \text{area}\,OAB = n \times \tfrac{1}{2}AO.OB \sin AOB = \tfrac{1}{2}nR^2 \sin \frac{2\pi}{n},$$

and, in terms of r, $= n \times AC.OC = nr^2 \tan \dfrac{\pi}{n}$.

If the circle be the same in each case, then the sides of its inscribed and circumscribed polygons will be $2r \sin \dfrac{\pi}{n}$, $2r \tan \dfrac{\pi}{n}$, and their areas $\tfrac{1}{2}nr^2 \sin \dfrac{2\pi}{n}$, $nr^2 \tan \dfrac{\pi}{n}$, which may be written $nr^2 \sin \dfrac{\pi}{n} \times \cos \dfrac{\pi}{n}$, $nr^2 \sin \dfrac{\pi}{n} \div \cos \dfrac{\pi}{n}$.

119. Hence also we may deduce the area of a circle in terms of its radius.

For, as above, the area of a regular polygon of n sides $= nr^2 \tan \dfrac{\pi}{n} = \pi r^2 \times \tan \dfrac{\pi}{n} \div \dfrac{\pi}{n}$: where, by increasing n, the number of sides, the polygon tends continually more and more to equality with the circle, radius r, and the ratio $\tan \dfrac{\pi}{n} \div \dfrac{\pi}{n}$ by (91) tends to 1: hence, as n is increased indefinitely, the Limit of the geometrical figure is the circle, radius r, and the Limit of the corresponding algebraical expression for its area is πr^2; hence, making n infinite, we have the area of the circle, radius r, $= \pi r^2$, as in (25).

Ex. 21.

1. Shew what the expressions for R and r in terms of the sides become, when the triangle is equilateral, or $a = b = c$.

2. Obtain the same when the triangle is isosceles, or $a = b$.

3. In the figure of (115), shew that the products of the alternate segments of the sides are each equal to $\frac{1}{2}(a + b + c)\, r^2$.

4. Obtain expressions for AC and BD in the figure of (116); and apply them to shew that $AC.BD = AB.CD + AD.BC$, and that $AC : BD :: AB.BC + AD.DC : BA.AD + BC.CD$.

5. Let $AB\ (2A)$ and $BC\ (2B)$ be two circular arcs: draw the diameter BD, and apply the first of the two results in (4), to deduce the expression for $\sin(A+B)$ in terms of A and B.

6. If the diagonals of *any* quadrilateral $ABCD$ intersect in E, shew that its area $= \frac{1}{2}AC.BD \sin E$; and thence that in the quadrilateral of (116) $\sin E = \dfrac{2\sqrt{\{(S-a)(S-b)(S-c)(S-d)\}}}{ac+bd}$.

7. Express the radius of the inscribed circle of a triangle in terms of the radii of its three escribed circles.

8. If R, r, be the radii of the circumscribed and inscribed circles of a triangle, and r_1 of the escribed circle touching the side a, express r and r_1 in terms of R and the angles, and shew that $r = r_1 \tan\frac{1}{2}B \tan\frac{1}{2}C$.

9. Shew that $R = \frac{1}{2}\sqrt[3]{\dfrac{abc}{\sin A \sin B \sin C}}$, and $2Rr = \dfrac{abc}{a+b+c}$.

10. If the triangle be right-angled at C, shew that

$$R + r = \tfrac{1}{2}(a+b).$$

11. If P, Q, be the centres of the circumscribed and inscribed circles of a triangle, shew that

$$\angle PAQ = \tfrac{1}{2}(B \sim C),\ \angle PBQ = \tfrac{1}{2}(A \sim C),\ \angle PCQ = \tfrac{1}{2}(A \sim B).$$

12. If the sides of the quadrilateral in (116), taken in order, be 3, 3, 4, 4, find its area, and the radii of the inscribed and circumscribing circles.

CHAPTER VI.

ON THE SOLUTION OF TRIANGLES.

We now proceed to the general solution of triangles by the aid of Trigonometry.

120. If the triangle be *right-angled*, we shall have here *one* datum already, namely, that one of the angles, C suppose, $= 90°$, and we require only two other parts to be given, in order to solve the triangle completely, one of these, however, (112) being a *side*.

Hence *four* cases may occur.

(1) Given a, A, or *one side and one angle.*

$$\text{Here } B = 90° - A, \quad b = a \cot A \ \text{ or } = \frac{a}{\tan A},$$

$$c = a \operatorname{cosec} A \text{ or } = \frac{a}{\sin A};$$

so that, in logarithms,

$$\begin{aligned} \log b &= \log a + \log \cot A &&= \log a + \text{L} \cot A - 10, \\ \text{or} \quad &= \log a - \log \tan A &&= \log a - \text{L} \tan A + 10; \\ \log c &= \log a + \log \operatorname{cosec} A &&= \log a + \text{L} \operatorname{cosec} A - 10, \\ \text{or} \quad &= \log a - \log \sin A &&= \log a - \text{L} \sin A + 10. \end{aligned}$$

This case includes, of course, that in which B, b are given, or in which A, b, or B, a, are given; since, when one of the angles A or B is given, the other is given also: and similar remarks apply to the other cases which follow.

(2) Given c, A, or *the hypothenuse and an angle.*

$$\text{Here } B = 90° - A, \quad a = c \sin A, \quad b = c \cos A;$$

so that, in logarithms,

$$\log a = \log c + \text{L} \sin A - 10, \quad \log b = \log c + \text{L} \cos A - 10.$$

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(3) Given a, b, or *two sides.*

Here $\tan A = \frac{a}{b}$, or $L \tan A = 10 + \log a - \log b$;

then, having found A, we have

$$B = 90^\circ - A, \text{ and } c = a \operatorname{cosec} A \text{ or } = b \sec A.$$

We might have found c from its value $\sqrt{(a^2 + b^2)}$; but, unless a and b should be numbers of one or two figures only, this would be more laborious, as the above expression is not adapted to logarithmic computation.

(4) Given a, c, or *one side and the hypothenuse.*

Here $\sin A = \frac{a}{c}$, or $L \sin A = 10 + \log a - \log c$;

and now, as before,

$$B = 90^\circ - A, \quad b = c \cos A \text{ or } = a \cot A.$$

Here also $b = \sqrt{(c^2 - a^2)} = \sqrt{\{(c + a)(c - a)\}}$;

$$\therefore \log b = \tfrac{1}{2} \log(c + a) + \tfrac{1}{2} \log(c - a).$$

Ex. 22.

Verify each of the above methods upon the following triangle:

$a = 123.4567$ $\log a = 2.0915147$ $\log(c + a) = 2.5894249$

$b = 234.5678$ $\log b = 2.3702685$ $\log(c - a) = 2.1511121$

$c = 265.0721$ $\log c = 2.4233642$ $A = 27^\circ\, 45'\, 31''$

$L \sin A = 9.6681505$, $L \cos A = 9.9469043$, $L \tan A = 9.7212462$.

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121. If the triangle be *oblique*, there will also be four cases, one side at least being given in each.

(1) Given a, A, B, or *one side and two angles.*

Here $C = 180^\circ - (A + B)$, $b = a \dfrac{\sin B}{\sin A}$, $c = a \dfrac{\sin C}{\sin A}$;

so that, in logarithms,

$$\log b = \log a + L \sin B - L \sin A, \quad \log c = \log a + L \sin C - L \sin A.$$

This, of course, includes the case when a, A, C, are given, and also the case when a, B, C, are given, since, if B, C, are given, A is also given.

(2) Given a, b, A, or *two sides, and an angle opposite to one of them.*

Here $\sin B = \frac{b}{a} \sin A$, or $\text{L} \sin B = \text{L} \sin A + \log b - \log a$;

and now $C = 180° - (A + B)$, $c = a \frac{\sin C}{\sin A}$.

Here, however, if $a < b$, we light upon the ambiguous case (112); and this ambiguity appears also from the above algebraical expression for $\sin B$. For, if $a < b$, then $A < B$; and, therefore, since A must be $< 90°$, (otherwise, we should have $A + B > 180°$,) B may be either greater or less than 90°, that is, B may be either of the two angles whose sine $= \frac{b}{a} \sin A$, one of them being, of course, supplementary to the other.

If, however, $a > b$, then $A > B$, and B *must* be $< 90°$; so that we have here no uncertainty, but must take for B the *lesser* of the two angles whose sine $= \frac{b}{a} \sin A$.

Ex. 23.

Verify each of the above methods upon the following triangle:

$A=44°$ $a=765.4321$ $\text{L} \sin A = 9.8417713$ $\log a = 2.8839067$
$B=66°$ $b=1006.62$ $\text{L} \sin B = 9.9607302$ $\log b = 3.0028656$
$C=70°$ $c=1035.43$ $\text{L} \sin C = 9.9729858$ $\log c = 3.0151212$
and, for the amb. case, $\text{L} \sin 22° = 9.5735754$, $\log 412.7725 = 2.6157108$.

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122. (3) Given a, b, C, or *two sides and the included angle.*

Let a be the greater of the two given sides.

Then $\frac{a}{b} = \frac{\sin A}{\sin B}$, whence (Alg. Part I. 85)

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} \text{ by (81):}$$

hence $\tan\frac{1}{2}(A-B) = \dfrac{a-b}{a+b}\tan\frac{1}{2}(A+B) = \dfrac{a-b}{a+b}\cot\frac{1}{2}C,$

$$\text{since } \tfrac{1}{2}(A+B) = \tfrac{1}{2}(180^\circ - C) = 90^\circ - \tfrac{1}{2}C;$$

$$\therefore \text{L}\tan\tfrac{1}{2}(A-B) = \text{L}\cot\tfrac{1}{2}C + \log(a-b) - \log(a+b),$$
$$= 20 - \text{L}\tan\tfrac{1}{2}C + \log(a-b) - \log(a+b);$$

and $\frac{1}{2}(A-B)$ is, therefore, known:

whence $A = \frac{1}{2}(A+B) + \frac{1}{2}(A-B)$, and $B = \frac{1}{2}(A+B) - \frac{1}{2}(A-B)$,

$$\text{and then } c = a\frac{\sin C}{\sin A} \text{ or } = b\frac{\sin C}{\sin B}.$$

123. We may, however, obtain a different expression for c as follows:

$$\frac{c}{a+b} = \frac{\sin C}{\sin A + \sin B} = \frac{\sin(A+B)}{\sin A + \sin B} = \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)};$$

$$\therefore c = (a+b)\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} = (a+b)\frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}(A-B)}.$$

Or otherwise: we have

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= (a^2+b^2)(\cos^2\tfrac{1}{2}C + \sin^2\tfrac{1}{2}C) - 2ab(\cos^2\tfrac{1}{2}C - \sin^2\tfrac{1}{2}C) \\ &= (a+b)^2\sin^2\tfrac{1}{2}C + (a-b)^2\cos^2\tfrac{1}{2}C \\ &= (a+b)^2\sin^2\tfrac{1}{2}C\left\{1 + \left(\frac{a-b}{a+b}\cot\tfrac{1}{2}C\right)^2\right\}. \end{aligned}$$

Now, since the tangent of an angle may be of any sign or magnitude,

$$\text{put } \tan\phi = \frac{a-b}{a+b}\cot\tfrac{1}{2}C;$$

then $c^2 = (a+b)^2\sin^2\frac{1}{2}C(1+\tan^2\phi) = (a+b)^2\sin^2\frac{1}{2}C\sec^2\phi,$

$$\text{or } c = (a+b)\sin\tfrac{1}{2}C\sec\phi = (a+b)\frac{\sin\frac{1}{2}C}{\cos\phi},$$

which result, however, coincides with the former, since

$$\tan\phi = \frac{a-b}{a+b}\cot\tfrac{1}{2}C = \frac{\sin A - \sin B}{\sin A + \sin B}\tan\tfrac{1}{2}(A+B) = \tan\tfrac{1}{2}(A-B),$$

as appears by (81).

124. (4) Given a, b, c, or *the three sides.*

Here we must employ the formulæ in (107), for $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$, or $\tan\frac{1}{2}A$, each of which requires only *four* logarithms, whereas that for $\sin A$ requires *seven*, and is, therefore, less convenient for use. If *two* angles A, B, are to be found, it will be best to use the formulæ for $\tan\frac{1}{2}A$, $\tan\frac{1}{2}B$, as the same logarithms are required for each of them, which would not be the case if we used the other formulæ. In other cases any one of the three formulæ may be used; except that, when $\frac{1}{2}A$ nearly equals 90°, it is better not to determine it by the sine or tangent, nor when near zero by the cosine; inasmuch as (104) we should not be able, in such cases, to determine $\frac{1}{2}A$ with sufficient accuracy.

Ex. 24.

Verify the methods (3) and (4) on the triangle in Ex. **23**, having also given

$\text{L}\tan 35° = 9.8452268$ $\log(b+a) = 3.2484765$
$\text{L}\tan 11° = 9.2886522$ $\log(b-a) = 2.3823555$
$\text{L}\tan 22° = 9.6064096$ $\log s = 3.1473150$
$\log(s-a) = 2.8050649$, $\log(s-b) = 2.5989515$, $\log(s-e) = 2.5662477$.

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125. The angle ϕ, introduced in (123), is called a *subsidiary* angle. Such an angle may often be employed, as there, in order to throw an expression into *factors*, and so render it suitable for logarithmic computation.

The following are other instances of the use of such angles.

Ex. 1. To adapt $\sqrt{(a \pm b)}$ to logarithmic computation.

Here $\sqrt{(a \pm b)} = \sqrt{a}\,\sqrt{(1 \pm \frac{b}{a})} = \sqrt{a}\sec\theta$ or $\sqrt{a}\cos\theta$, if, for the upper sign, we put $\tan^2\theta = \frac{b}{a}$, and, for the lower sign, $\sin^2\theta = \frac{b}{a}$, in which latter case, however, b must be less than a, or we could not make the assumption, nor would the given expression $\sqrt{(a-b)}$ be a *possible quantity.*

Ex. 2. To adapt $a+b+c$ to logarithmic computation.

Here $a+b=a(1+\tan^2\theta)=a\sec^2\theta$, if $\tan^2\theta=\dfrac{b}{a}$;

and $a+b+c=a\sec^2\theta+c=a\sec^2\theta(1+\tan^2\phi)=a\sec^2\theta\sec^2\phi$,

$$\text{if } \tan^2\phi=\frac{c}{a\sec^2\theta}=\frac{c}{a}\cos^2\theta.$$

Ex. 3. To adapt $a\cos\alpha\pm b\sin\alpha$ to logarithmic computation.

Here $a\cos\alpha\pm b\sin\alpha=k\sin(\theta\pm\alpha)$, if $k\sin\theta=a$, $k\cos\theta=b$;

whence $k^2(\sin^2\theta+\cos^2\theta)=k^2=a^2+b^2$, or $k=\sqrt{(a^2+b^2)}$,

$$\text{and } \frac{k\sin\theta}{k\cos\theta}=\tan\theta=\frac{a}{b}.$$

The expression for k, however, may be adapted to logarithmic computation, for $k=\sqrt{(a^2+b^2)}=a\sqrt{(1+\cot^2\theta)}=a\operatorname{cosec}\theta$, a result which we obtain immediately from $k\sin\theta=a$.

These examples are sufficient to shew the method of proceeding in most ordinary cases. Thus, if we had to adapt to logarithmic computation such an expression as

$$a+b\cos\theta+c\cos(\theta-\alpha)+d\sin(\theta+\beta):$$

this may be written

$$a+(b+c\cos\alpha+d\sin\beta)\cos\theta+(c\sin\alpha+d\cos\beta)\sin\theta;$$

which, by Ex. 2, may be converted to $a+b'\cos\theta+c'\sin\theta$, (where b', c', will have been found in forms adapted to log. computation;) this again, by Ex. 3, may be expressed as $a+k\sin(\theta+\gamma)$, which last may be converted by Ex. 1 to $a\sec^2\phi$, if $\tan^2\phi=\dfrac{k}{a}\sin(\theta+\gamma)$.

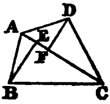
CHAPTER VII.

ON THE TRIGONOMETRICAL MEASUREMENT OF HEIGHTS AND DISTANCES.

126. We have already seen some instances of the application of Trigonometry to the actual measurement of heights and distances. We shall devote this chapter to illustrate the practical application of the Science to such cases as may commonly occur of a similar kind—that is, where the distance required cannot be *measured*, either on account of its great magnitude or of some intervening obstacle, and cannot also be more simply obtained by the ordinary rules of common Geometry. For, in actual surveys of land or countries, there are many results which may be obtained very easily without the help of Trigonometry; and of these we shall first give some instances.

127. Besides the property proved in *Euc.* I. 47, namely, that *the square upon the hypothenuse in a right-angled triangle is equal to the sum of the squares upon the two sides*, and the proposition deduced in (25) that the *area of any triangle* = ½ *base* × *height*, it is very useful to notice the following geometrical properties, which are much employed in common field-surveying.

(1) The area of any quadrilateral $ABCD$

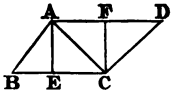


$$= \text{area } ABC + \text{area } ADC$$

$$= \tfrac{1}{2}AC \times BE + \tfrac{1}{2}AC \times DF = \tfrac{1}{2}AC(BE + DF),$$

if BE, DF, be the perpendiculars from the angles B, D, on the diagonal AC.

(2) The area of a quadrilateral $ABCD$, two of whose sides AD, BC, are parallel,



$= \text{area } ABC + \text{area } ADC$

$= \frac{1}{2}BC \times AE + \frac{1}{2}AD \times CF = \frac{1}{2}(BC + AD)AE$

$= \frac{1}{2}$(sum of parallel sides)×(perp. distance between them).

(3) The area of any quadrilateral $ABCD$

$= \text{area } ABE + \text{area } BEFC + \text{area } CFD$,

(if BE, CF, be perpendiculars on AD,)

$= \frac{1}{2}AE \times BE + \frac{1}{2}(BE + CF)EF$ by (2) $+ \frac{1}{2}CF \times FD$

$= \frac{1}{2}(AE + EF)BE + \frac{1}{2}(EF + FD)CF = \frac{1}{2}AF \times BE + \frac{1}{2}DE \times CF.$

N.B. The same result will be found to hold good, if the perpendiculars, either or both of them, should fall *without* the side AD: thus, if CF fall beyond D, (the Student may draw the figure,) we shall have area $ABCD$ = area ABE + area $BEFC$ − area CFD

$= \frac{1}{2}AE \times BE + \frac{1}{2}(BE + CF)EF - \frac{1}{2}CF \times FD$

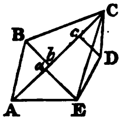
$= \frac{1}{2}(AE + EF)BE + \frac{1}{2}(EF - FD)CF$

$= \frac{1}{2}AF \times BE + \frac{1}{2}DE \times CF$, as before.

128. By means of the above properties, fields &c. are usually measured, being broken up into triangles or quadrilaterals, for which the necessary lines are measured, and the areas found as above.

The following Example will give an idea of the mode of proceeding in such a case, and the way in which the operation is conducted.

Let $ABCDE$ be a five-sided field, AC its longest diagonal: the surveyor measures along from A till he comes to a, where he sees the angle E in a direction at right angles to AC: he measures aE, and then measures on ab in the same line as before, to the point b, where B appears in a direction at right angles to AC; in like manner he measures bB, bc, cD, and finally cC. The line AC, joining the two extreme stations, is called the *chain-line*, and the perpendiculars aE, bB, cD, are called *offsets*.



The lengths of these lines are measured by *Gunter's Chain, which is 22 yards* or 4 poles long, and divided into 100 links.

Hence an acre, which contains 4840 square yards, will contain 10 square chains or 100,000 square links: and, therefore, square links are converted into acres by merely cutting off five figures to the right.

The perpendicularity of the lines aE, &c. may be ascertained by the use of the *cross-staff*, which is a circular piece of wood, about 6 inches across, on the face of which two slits are made, along diameters at right angles to one another. It is supported upon a vertical staff, which is thrust into the ground when the surveyor reaches a point, as at a, where the angle E *appears* to lie at right angles to the line AC. Then, the circle being so turned that one slit is in the direction AC, an eye will look along the other in the perpendicular direction, and the staff must be moved, if necessary, along AC, until the angle E can thus be seen.

Suppose now $Aa=480$, $aE=480$, $Ab=660$, $bB=200$, $Ac=860$, $cD=460$, and $AC=1340$ links. Then these results would be thus entered in the *field-book*, beginning from the bottom, the measures on the Chain Line being set down in the middle, and the offsets to the right or left on each side.

Left	Chain	Right
	1340 to C	
	860	460
200	660	
	480	480
	from A	

Hence the area of the field

$= \text{area}\,ABC + \text{area}\,AEDC = \frac{1}{2}AC \times bB + \frac{1}{2}Ac \times aE + \frac{1}{2}Ca \times cD$ [127 (3)]

$= \frac{1}{2}\{1340 \times 200 + 860 \times 480 + 860 \times 460\}$ *sq. links* = 538200 *sq. links*

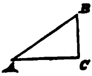
= 5.382 *acres* = 5A 1R 21P nearly.

129. Without entering further into the details of Land-surveying, which may be found in books specially devoted to the subject, we shall now give a few instances to illustrate the applications of Trigonometry.

It must be taken for granted that it is always possible, by means of proper instruments, to measure the angle subtended at the eye between two visible points, (that is, the angle between the lines joining the eye with the two points,) and also the angle at which an object appears *elevated* above, or *depressed* below, the horizontal plane through the eye of the observer. And it will be remembered that, in order to determine any triangle, *we must have three of its parts given, one of them being a side.* As a general rule then, in order to determine any particular side or angle which may be required in a Problem, we must first consider, if it be not a part of some triangle, for which we have already the three necessary *data;* and, if not, we must seek to determine, by means of the given data, three such parts for some triangle, to which the line or angle in question belongs.

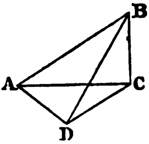
Ex. 1. *To find the height of a visible object, whose foot is accessible.*

This problem we have solved in some simple cases in (66): more generally, having measured a horizontal base, AC, and observed the angle of elevation BAC, then, in the right-angled triangle BAC, we have the two parts AC and A, and the height $BC = AC \tan A$.



Ex. 2. *To find the height of a visible object, whose foot is inaccessible, and its distance from the place of observation.*

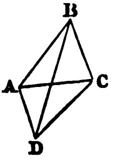
From A, the place of observation, measure a base AD in any convenient direction; and at each end of the base observe the angles which the base makes with a line drawn to the object, that is, observe the angles BAD, BDA.



Then, in the triangle BAD, we have the three necessary data, from which the *distance* AB may be determined; and, by observing also the angle of elevation, BAC, we have the height $BC = AB \sin A$.

Ex. 3. *To find the distance between any two visible, but inaccessible, objects, and the distance of either from the place of observation.*

Let B, C, be the two objects, A the place of observation; measure a base AD in any direction, but so that the two points, B, C, may both be visible from D; and observe the angles BAC, BAD, CAD, BDA, CDA.

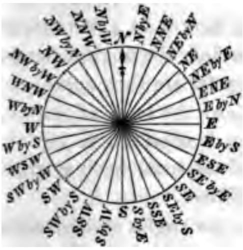


Then, in the triangle ABD, we have given AD, and angles BAD, BDA; hence we may determine the side AB: in the triangle ACD, we have given AD, and angles CAD, CDA; hence we may determine the side

AC: and now, in the triangle *BAC*, we have given *AB*, *AC*, and angle *BAC*; hence we may determine the side *BC*.

If *BC* be supposed to be vertical, we may in this way determine the *height* *BC*, when the line *AC* is not horizontal.

130. As the bearings by the Mariner's Compass are often referred to in Trigonometrical Problems, it is well that the Student should make himself acquainted with the lines of the Compass, and the names given to the different *Points*. It will be seen that there are in all 32 Points, the angle between two adjacent points being $\frac{360^\circ}{32} = 11\frac{1}{4}^\circ$.



Sometimes, however, the bearings of objects are referred to in Problems without the use of the Points, as so many degrees from the North (or South) towards the East (or West).

Thus if to a spectator P one object *A* appears to the NE and another *B* to the SSW, the angle between the lines drawn from his position to the two objects, that is, the angle *APB* would be 10 points = $112\frac{1}{2}^\circ$; and *A* may be said to have a bearing N 45° E, *B* a bearing N $157\frac{1}{2}^\circ$ E, or, better, S $22\frac{1}{2}^\circ$ W, reckoning from the *nearest* of the two points, N and S.

N.B. The logarithms required in the following Problems, (besides those which may be *given* in any of them,) will be found on page 112, or may be obtained from those there printed.

Ex. 25.

1. Calculate the area of a field from the annexed extract from a field-book.

	750 to *C*	
420	540	
	125	280
	from *A*	

2. A river *AC*, whose breadth is 200 feet, runs at the foot of a tower *BC*, which subtends an angle *BAC* of 25° 10′ at the edge of the bank. Required the height of the tower, having given Log tan 25° 10′ = 9.6719628, log 9.397 = .9729928.

3. Two observers, 1000 yards apart, in the same vertical plane with a balloon, but on opposite sides of it, take its angles of *elevation at the same* moment, 36° and 54°: determine its height.

4. A person observes the elevation of a tower to be 60°, and, on retiring from it 100 yards further, he finds the elevation to be 30°: determine the height of the tower.

5. Find the area of a four-sided field from the annexed extract.

	1045 to *B*
435	710
147	225
	from *A*

6. Two men are surveying, and, when each is at a distance of 200 yards from the flag-staff, one of them finds the angle between it and the other's position to be 36°. How far are they apart?

7. From the top of a rock, 500 feet above the level of the sea, the angle of depression of a ship's bottom is observed to be 18°: find the ship's distance from the foot of the rock, having given $\text{Log}\tan 18^\circ = 9.5117760$, $\log 1.5388 = .1871940$.

8. At the top of a tower, 108 feet high, the angles of depression of the top and bottom of an upright column, standing on the same horizontal plane with the tower, were found to be 30° and 60°: find its height.

9. In a right-angled field, given one side to be 75 yards, and the perpendicular from the right angle upon the opposite side to be 60 yards, determine the other sides of the triangle and the area of the field.

10. To determine the distance of a ship at anchor at C, I measured a straight line AB of 1000 yards along the shore, and observed the angles $CAB = 32^\circ\,10'$, $CBA = 83^\circ\,10'$. Find the ship's distance from A, having given $\text{Log}\cos 6^\circ\,50' = 9.9969040$, $\text{Log}\cos 25^\circ\,20' = 9.9560886$, $\log 1.0985 = .0408154$.

11. Determine the sides and area of a right-angled field, when one angle is 60°, and the line joining the right angle with the middle point of the opposite side is 100 yards.

12. At the top of a pillar, 100 feet high, the elevation of a tower is observed to be 30°, and at the foot of the pillar it is 60°. Find the height of the tower.

13. A is a point invisible to an observer at B: shew that he may find the distance AB by measuring a base BC (300 yards), and observing the angles BCA (30°) and BAC (54°).

14. The elevation of an object, standing upon a horizontal plane, is observed, and, 80 feet nearer, the elevation is found to be the complement of the former, and now, on retiring 30 feet, it *is double of its original* value. Find the height of the object.

15. A light-house was observed from a ship to bear N 45° E, and, after sailing due South for 6 miles, it bore N 30° E. Find its distance from the ship at each observation.

16. Find the area of a four-sided field, two of whose sides are parallel, their lengths 230 and 480 yards, and their distance 200 yards.

17. The length of a road, in which the ascent is 1 foot in 5, is $1\frac{3}{8}$ mile, from the foot of the hill to the top. What will be the length of a road up the same hill, in which the ascent shall be 1 foot in 12?

18. Find the area of a trapezium, the diagonal being $108\frac{1}{2}$ feet, and the perpendiculars on it $56\frac{1}{4}$ feet and $60\frac{3}{4}$ feet, respectively.

19. A person, standing on a river's bank, observes the elevation (a) of a tower on the opposite bank: going backwards a feet, he then finds the elevation to be $\frac{1}{2}a$. Determine the height of the tower, and the breadth of the stream.

20. An object B is invisible from A: but a base AC (80 feet) is measured, and the angle ACB (144°) observed; and again, at D in the same line, 40 feet further on, the angle ADB is found to be the supplement of ACB. Find the distance AB.

21. From the top of a hill I observe two objects, in a straight horizontal line, directly before me, and find their angles of depression to be 45° and 30° respectively. I know that the objects are $\frac{1}{10}$ of a mile apart. What shall I find to be the height of the hill?

22. Two objects, P and Q, were observed from a ship to be at the same instant in a line bearing N 15° E. After sailing NW for 5 miles, P bore due E, and Q bore NE. Find the distance between P and Q.

23. B is a point which cannot be seen from A: but a surveyor measures a line AC (300 yards) and, going on still in the same direction, he measures CD (100 yards), and observes the angles ACB (54°) and ADB (36°). Determine AB.

24. The elevation of a tower, 100 feet high, when due N of an observer, was 60°: what will it be after he has walked due E 400 feet, given $\tan 13° 53' = .2471663$, $\tan 13° 54' = .2474750$?

CHAPTER VIII.

ON LOGARITHMS, AND THE EXPONENTIAL THEOREM.

131. Def. The *logarithm* of a number to a given base is *the index of that power to which the base must be raised, in order to become equal to the number:* so that, if a^x = any number N, then x is called the *logarithm of N to the base a*, which is denoted thus, $x = \log_a N$, or (if there be no occasion to mention the base) $x = \log N$.

Thus, since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, &c., $\therefore \log_{10} 1 = 0$, $\log_{10} 10 = 1$, $\log_{10} 100 = 2$, &c.; and so also, since $a^0 = 1$, $a^1 = a$, we have $\log_a 1 = 0$, $\log_a a = 1$, whatever a may be.

132. By taking any *positive* number (except unity) for base, we may express any *positive* number as some power of it.

Thus, take $a = 10$, as above, and let $N = 2$; then, since $10^0 = 1$ and $10^1 = 10$, there is some value of x, if we could find it, between 0 and 1, such that $10^x = 2$: and, in point of fact, it may be shewn that this value is (to five places of decimals) .30103, so that $\log_{10} 2 = .30103$. It would of course be possible, though tedious, to verify this statement by expanding $10^{.30103}$ or $(1+9)^{\frac{30103}{100000}}$ to a sufficient number of terms by the Binomial Theorem: but we shall see below that such would certainly be the case.

133. If we give to N the successive values 1, 2, 3, &c., and register the corresponding values of x, (which may be found by methods to be given hereafter,) the table thus formed is called a 'Table of Logarithms to the base 10'.

The integral part of any logarithm is called the *Characteristic*, the decimal part is called the *mantissa*, or *handful*, as it were, thrown in over and above the characteristic. Of course when there is no integral part, the characteristic is 0.

134. If the base a be > 1, then since $a^{-\infty} = 0$, $a^0 = 1$, $a^\infty = \infty$, we have $\log_a 0 = -\infty$, $\log_a 1 = 0$, $\log_a \infty = \infty$; and thus the log. of any number between 0 and 1 will lie between 0 and $+\infty$, and the log. of any number between 1 and ∞ will lie between 0 and $-\infty$; that is, the log. of a number will be positive or negative, according as the number itself is $>$ or < 1. Of course, the contrary will be the case if we suppose a to be < 1.

135. Although, in reasoning generally about Logarithms, we may consider the base to be any positive number other than unity, yet in actual practice we shall have only to deal with (i) Logarithms to the base 10, which are called *Common* Logarithms, and (ii) Logarithms to the base e, where e denotes a certain number 2.7182818, (of which more hereafter,) which are called *Napierian* Logarithms, from the name of Lord Napier of Merchiston in Scotland, who first invented Logarithms.

136. It will be seen below that series may be found, by means of which logarithms to the base e may be readily calculated. Among these will be

$$\log_e 10 = 2.3025850928;$$

and we shall now shew that, if the logarithm of any number to base e be multiplied by the factor

$$1 \div \log_e 10 = 1 \div 2.302\&c. = .4342944,$$

it will become the corresponding logarithm to the base 10.

For let x, y, be the logarithms of any number N to any two bases a and b, that is, let $x = \log_a N$, $y = \log_b N$, and $\therefore a^x = N = b^y$: then, since (131) $b = a^{\log_a b}$, we have

$$a^x = (a^{\log_a b})^y = a^{\log_a b.y},$$

and $\therefore x = \log_a b.y$, or $\log_a N = \log_a b.\log_b N$.

Hence, of course, it follows that

$$\log_e N = \log_e 10 . \log_{10} N, \text{ or } \log_{10} N = \frac{1}{\log_e 10} \times \log_e N.$$

This constant multiplier, by which the two systems of logarithms are connected, is called the *Modulus* of the system to base 10, and will be denoted in future by M.

Cor. $\text{Log}_a N = \log_a b . \log_b N = \log_a b . \log_b c . \log_c N =$ &c., and so on, through any number of bases.

137. We will suppose then that, by means of the Napierian logarithms, we have formed a Table of Common Logarithms, and from these we will extract for our present use the following.

Logarithms, to base 10, *of all Prime Numbers from* 1 *to* 100.

No.	Logs.	No.	Logs.	No.	Logs.	No.	Logs.
2	0.3010300	19	1.2787536	43	1.6334685	71	1.8512583
3	0.4771213	23	1.3617278	47	1.6720979	73	1.8633229
7	0.8450980	29	1.4623980	53	1.7242759	79	1.8976271
11	1.0413927	31	1.4913617	59	1.7708520	83	1.9190781
13	1.1139434	37	1.5682017	61	1.7853298	89	1.9493900
17	1.2304489	41	1.6127839	67	1.8260748	97	1.9867717

138. Now the following are the properties of logarithms which make them of singular value in diminishing the labour of Arithmetical calculations.

(i) $\text{Log}(mn) = \log m + \log n$, or the logarithm of any product = the sum of the logarithms of the factors.

For, if $m = a^x$, and $n = a^y$, then $mn = a^{x+y}$, and

$$\therefore \log(mn) = x + y = \log m + \log n.$$

Hence also

$$\log(mnp) = \log(mn) + \log p = \log m + \log n + \log p, \text{ \&c.}$$

(ii) $\text{Log}\frac{m}{n} = \log m - \log n$, or the logarithm of any quotient = the remainder obtained by subtracting the logarithm of the divisor from that of the dividend.

For $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$, and $\therefore \log(\frac{m}{n}) = x - y = \log m - \log n$.

Hence $\log \frac{1}{m} = \log 1 - \log m = -\log m$, since $\log 1 = 0$.

(iii) $\text{Log}\, m^n = n \log m$, or the logarithm of any power of a number is obtained by multiplying the logarithm of the number by the index of the power.

For $m^n = (a^x)^n = a^{nx}$, and $\therefore \log(m^n) = nx = n \log m$.

As the index in (iii) may be either *integral* or *fractional*, we see that, by means of the above results, all Arithmetical operations of Multiplication, Division, Involution, and Evolution, may be converted into *Addition* and *Subtraction* of Logarithms.

Ex. 1. $\log 6 = \log 2 + \log 3 = .7781513$;
$\log 5 = \log 10 - \log 2 = 1 - \log 2 = .6989700$.

Ex. 2. $\log 100 = \log 10^2 = 2 \log 10 = 2$, $\log 1000 = 3$, &c.

Ex. 3. $\log 4 = \log 2^2 = 2 \log 2 = .6020600$;
$\log 18 = \log 2 + \log 9 = \log 2 + 2 \log 3 = 1.2552725$.

Ex. 4. $\log .07 = \log \frac{7}{100} = \log 7 - \log 100 = .8450980 - 2$, which is written thus, $\bar{2}.8450980$, it being understood that, in this position of the negative sign, it belongs only to the characteristic 2, and not to the mantissa, which is still positive.

Ex. 5. $\log 2.4 = \log \frac{24}{10} = \log 3 + \log 8 - 1 = .4771213 + .9030900 - 1$
$= .3802113$;
$\log .0023 = \log \frac{23}{10000} = 1.3617278 - 4 = .3617278 - 3 = \bar{3}.3617278$.

Ex. 6. $\log \frac{3}{97} = .4771213 - 1.9867717 = -1.5096504$, which may be written thus, $-2 + (1 - .5096504) = \bar{2}.4903496$.

139. It will be seen from [138 Ex. 6] how we may readily convert a logarithm which is *wholly* negative into one that shall be negative only in its characteristic, viz. *by increasing the given negative characteristic by unity, and subtracting from unity the given mantissa.*

Thus, generally, if $-(C+m)$ represent a logarithm in which the characteristic (C) and mantissa (m) are both negative, then

$$-(C+m) = -C-m = -(1+C)+(1-m),$$

where the mantissa is now positive.

The decimal thus obtained by subtracting another from unity is called its *Arithmetical Complement*, and is most readily written down in practice by subtracting its last figure from 10 and the others from 9. A logarithm thus modified we may call a *Complementary* Logarithm, and denote by colog.

By the use of Complementary Logarithms the Subtraction of a logarithm may be turned into Addition.

Thus $\log \frac{3}{97} = \log 3 + \text{colog}\, 97 = .4771213 + \bar{2}.0132283 = \bar{2}.4903496$, as before.

But it is necessary to notice some peculiarities which occur in the use of such logarithms, whose characteristics only are negative, and of which instances are given in the following examples.

Ex. 1. $\log \frac{2}{17} + \log \frac{3}{19} = \log 2 + \text{colog}\, 17 + \log 3 + \text{colog}\, 19$

$$\begin{array}{r} = 0.3010300 \\ + \bar{2}.7695511 \\ + 0.4771213 \\ + \bar{2}.7212464 \\ \hline \bar{2}.2689488 \end{array}$$

Here there was an integer $+2$, arising from the sum of the positive mantissæ, which, combined with the sum of the characteristics -4, leaves -2 or $\bar{2}$.

Ex. 2. $\log \left(\frac{1}{13}\right)^3 = 3 \log \frac{1}{13} = 3\, \text{colog}\, 13 = \bar{2}.8860566$

$$\begin{array}{r} \bar{2}.8860566 \\ 3 \\ \hline \bar{4}.6581698 \end{array}$$

Here there was an integer $+2$, arising from the product of the positive mantissa by 3, which, combined with $3 \times \bar{2} = -6$, leaves -4 or $\bar{4}$.

Ex. 3. $\log \sqrt[7]{\frac{1}{71}} = \frac{1}{7}\, \text{colog}\, 71 = \frac{1}{7}(\bar{2}.1487417) = \bar{1}.7355345$.

Here it was necessary to imagine the logarithm thrown into the equivalent form $\frac{1}{7}(\bar{7} + 5.1487417)$, so that the negative characteristic may become a multiple of the divisor 7.

Ex. 26.

Obtain the logarithms of

1. 8, 9, 12.
2. 20, 25, 60.
3. $\frac{1}{3}, \frac{1}{4}, \frac{2}{5}$.
4. .03, $\frac{1}{30}$, .0033.
5. 1.8, 140, 1.44.
6. .0625, $\frac{1}{1210}$, 1.05.
7. 10.6, $4\frac{1}{5}$, $4\frac{3}{5}$.
8. .0111, $\sqrt{1\frac{1}{2}}$, $\sqrt[3]{1\frac{1}{8}}$.
9. $\sqrt{.1}$, $\sqrt[3]{.02}$, $(1.2)^{\frac{2}{3}}$.
10. $(\frac{1}{7})^3$, $\frac{3}{4}\sqrt{1.1}$, $(.069)^{\frac{2}{3}}$.
11. $\sqrt[4]{\frac{1}{2}} \times \sqrt[3]{\frac{2}{3}} \times \sqrt{\frac{3}{4}}$.
12. $\sqrt{\{\frac{1}{2}\sqrt[3]{(\frac{2}{3}\sqrt[4]{\frac{3}{4}})}\}}$.

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140. There are, however, two observations to be made, which greatly facilitate the finding of Common logarithms.

(i) In the Common system having given $\log N$, we can find immediately $\log(N \times 10^n)$ or $\log(N \div 10^n)$, n being an integer, that is, we can find the logarithm of any number, *which differs from N only in the position of the decimal point.*

For $\log(N \times 10^{\pm n}) = \log N \pm n \log 10 = \log N \pm n$, and consequently, (since n is an integer,) will differ from $\log N$ only in the characteristic, which will be increased or diminished by n, while both logarithms will have the same mantissa.

Thus, suppose that we have given $\log 1362 = 3.1341771$:

then $\log 136200 = \log(1362 \times 10^2) = \log 1362 + 2 = 5.1341771$,

$\log 1.362 = \log(1362 \div 10^3) = \log 1362 - 3 = .1341771$,

$\log .001362 = \log(1362 \div 10^6) = \log 1362 - 6 = \bar{3}.1341771$.

(ii) In the Common system, the characteristic of the logarithm of any number may be written down at once by inspection.

For if the number lie between 1 and 10, that is, between 10^0 and 10^1, its logarithm must lie between 0 and 1, and, therefore, its characteristic will be 0; so, if it lie between 10 and 100, that is, between 10^1 and 10^2, its characteristic will be 1; if it lie between 100 and 1000, that is, between 10^2 and 10^3, its characteristic will be 2;

and generally, if it have n digits, that is, if it l between 10^{n-1} and 10^n, its logarithm will lie betwe $n-1$ and n, and its characteristic will be $n-1$.

Again, if the number lie between 1 and .1, that between 1 and $\frac{1}{10}=10^{-1}$, its logarithm must lie betwe 0 and -1, and, therefore, (with positive mantissa) characteristic will be $\bar{1}$; so, if it lie between .1 and .0 that is, between 10^{-1} and 10^{-2}, the characteristic w be $\bar{2}$; if it lie between .01 and .001, that is, betwe 10^{-2} and 10^{-3}, the characteristic will be $\bar{3}$; and general if it have $n-1$ cyphers after the point, the characteris will be $\bar{n}$.

It will be seen that both cases are comprised in t following Rule: Reckon the distance of the first *s* *nificant figure* from the *units-place:* if it be n places the left, the characteristic will be $+n$; if n places the right, $\bar{n}$.

141. Hence it appears that, in the Tables of Comm Logarithms, it is only necessary to register the *m* *tissæ,* corresponding to certain *sequences* of figu which look like numbers, but are not really so, beca the place of the decimal point is not fixed in them.

Thus in the Table given below, we have, opposite to the v 3570 for N, the mantissa 5526682, where 3570 is no num but merely a sequence of figures, and the Table shews that all such sequences, wherever we insert in them the decimal po the mantissa is still 5526682: thus, determining the characteri in each case by [140 (ii)], we have

$$\log 3.570 \text{ or } \log 3.57 = .5526682, \quad \log 35700 = 4.5526682,$$
$$\log .00357 = \bar{3}.5526682.$$

142. The mantissæ of logarithms are registered *the best Tables* to 7 decimal places, corresponding

sequences of 5 figures, as in the lines below, extracted from Hutton's Tables.

N.	0	1	2	3	4	5	6	7	8	9	D.	Pro.
570	5526682	6804	6925	7047	7169	7290	7412	7534	7655	7777	122	122 1 12 2 24 3 37 4 49 5 61 6 73 7 85 8 98 9 110
71	7899	8020	8142	8263	8385	8507	8628	8750	8871	8993		
72	9115	9236	9358	9479	9601	9722	9844	9965	$\dot{0}$087	$\dot{0}$209		
73	5530330	0452	0573	0695	0816	0938	1059	1181	1302	1424		

Thus, to find the mantissa for the sequence $N = 35725$, we look horizontally along the third column for 3572, and vertically down under the figure 5, and thus find it to be 5529722, including the figures 552, which, being once printed (as in the first line) at the beginning of the line in which they first occur, are understood to be repeated before each mantissa, until (as in the fourth line) they are replaced by 553. This change, however, actually begins with the last two mantissæ of the third line, where the dotted figures are introduced to mark this: thus, the mantissa for 35728 is 5530087.

In some books, instead of a dotted figure, a figure with a line above it, as $\bar{0}$, or a *smaller* figure, is used to mark the point where this change begins, if it happen not to be at the beginning of a line.

143. Although the mantissæ are only given in the Tables for sequences of five figures, yet they may be readily found for sequences of six or seven figures by the following considerations.

It will be shewn hereafter, that when the difference of two numbers is small compared with either of them, the difference of their logarithms is very nearly proportional to the difference of the numbers.

Now let m, $m + D$, be the mantissæ for two consecutive numbers, N and $N + 1$, each of five figures, $m + \delta$ the mantissa for a number $N + d$, lying between them, that is, having the same integral part as N, but one or more figures after the decimal point. Then, since the three numbers have all the same number of integral digits, they will have all the same characteristic, and so the difference of their mantissæ will be the same as the difference of their logarithms: hence, by the above statement,

$$D : \delta :: 1 : d, \quad \text{or} \quad \delta = Dd,$$

by means of which result we may find δ when d is given, or, conversely, d when δ is given.

Ex. 1. To find log 35.7235.

Here $N = 35723$, $N + 1 = 35724$, $N + d = 35723.5$, and $\therefore d = \frac{5}{10}$; and $D =$ the Difference of the mantissæ of 35723 and $35724 = 122$: hence for 35723.5, $\delta = \frac{5}{10}$ of $122 = 5 \times 12.2 = 61$; and thus, the whole mantissa for 35723.5 being $5529479 + 61 = 5529540$, we have

$$\log 35.7235 = 1.5529540.$$

So for 35723.57 the difference (δ) is

$$\tfrac{57}{100} \text{ of } 122 = 57 \times 1.22 = 69.54 = 70 \text{ nearly},$$

and $\therefore$ $\log .003572357 = \bar{3}.5529549$.

Ex. 2. To find the number corresponding to the $\log \bar{2}.5528797$.

Here the next lower mantissa is that for 35717, viz. 5528750, and $\therefore$ $\delta = 47$: hence, since the difference (D) between the mantissæ for 35717 and 35718 is 121, we have $d = \delta \div D = \frac{47}{121} = .38$, (it being useless to go beyond two decimal places, for a reason that will appear hereafter;) and so if the *number* of five figures, referred to as N in the proof, be 35717, the *number* $N + d$ will be 35717.38; from which we see that the given mantissa corresponds [140] to the *sequence* 3571738, and therefore the given logarithm (the characteristic being $\bar{2}$) corresponds to the *number* .03571738.

144. But the columns headed D. and Pro. (Proportional Parts) are intended to facilitate the calculation of such logarithms, and the converse operation of finding the corresponding number from the given logarithm. The column D. shews that the *prevailing* difference between two consecutive mantissæ in this neighbourhood is 122, as will be seen at once to be the case by looking

at them, the difference being sometimes 121, but generally 122: further back in the Tables, it would have been 123, 124, &c., and further on, 121, 120, &c.

Now, for each value of D, there is formed what is called a Table of Proportional Parts, by multiplying $\frac{D}{10}$, that is, in this case, 12.2, by 1, 2, 3, &c., which products give 12.2=12 (nearly), 24.4=24, 36.6=37, &c., (as in the extract on page 117), the integer only being retained in each product, but (15 Ex. 2) increased by unity when the rejected decimal part equals or exceeds $\frac{1}{2}$ or .5. The use of this will now be apparent: for since $\delta = Dd$, let α, β, &c. be the figures in order of the decimal d; then

$$\delta = D\left(\frac{\alpha}{10} + \frac{\beta}{100} + \&c.\right) = \alpha\frac{D}{10} + \beta\frac{D}{100} + \&c.$$

Now, in the Table of Pro. Parts, the first column contains the successive values 1, 2, 3, &c. which α or β might have, and the second contains the corresponding values of $\alpha\frac{D}{10}$. In order, therefore, to find $\alpha\frac{D}{10}$, we have only to glance at the Table, and take down the number opposite to the value of α; and, in order to find $\beta\frac{D}{100}$, we have only to take down $\frac{1}{10}$th of the number, opposite to the value of β.

Ex. 1. To find log 357.2357.

Here we have mantissa for 35723 = 5529479
diff. for 5 = 61
diff. for 7 = 8.5 = 9
5529549

and, therefore, the logarithm required is 2.5529549.

Ex. 2. To find the number corresponding to the log., $\bar{1}.5529789$.

Here the mantissa next below the given one being that for *35725, viz. 5529722,* we have $\delta = 67$, from which taking 61, which we see, by the Table of Proportional Parts, gives the *sixth* figure 5, we have still remaining 6, which, being $\frac{1}{10}$ of 60, shews that the *seventh* figure is 4 (nearly 5); hence the *sequence* required is 3572554, and the *number* (the characteristic being $\bar{1}$) is .3572554.

Ex. 27.

In these Examples the Tables in pp. 112, 117, *are to be consulted.*

1. Find the logs of 35.70925, .3572739, .003571246, 3.572804.
2. Find the numbers whose logarithms are
   3.5528742, $\bar{1}.5530895$, $\bar{4}.5527777$, $.5530199$.
3. Given $\log 2000.1 = 3.3010517$, construct a Table of Pro. Parts; and find $\log 20.00094$, and the number whose log is $\bar{2}.3010489$.
4. Given $\log 31.001 = 1.4913757$,
   find the logarithms of 3100023, $31000\frac{1}{4}$, and $31\frac{3}{4000}$.
5. Find the value of $\sqrt{2\frac{1}{3}} \times \sqrt[3]{3\frac{1}{4}} \times \sqrt[4]{4\frac{1}{5}} \times \sqrt[5]{5\frac{1}{6}}$,
   given $\log 4.4985 = .6530677$, $\log 4498.6 = 3.6530774$.
6. Find the value of $\sqrt{357.1328} \div \sqrt[3]{35712.75}$,
   given $\log 57.386 = 1.7588060$, $\log .057387 = \bar{2}.7588135$.

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145. We shall now explain the method of calculating logarithms to the base e.

Exponential Theorem: To expand a^x in a series of ascending powers of x.

$$a^x = \{1 + (a-1)\}^x$$

$$= 1 + x(a-1) + \frac{x(x-1)}{1.2}(a-1)^2 + \frac{x(x-1)(x-2)}{1.2.3}(a-1)^3 + \&c.$$

$$= 1 + x\left\{(a-1) + \frac{x-1}{2}(a-1)^2 + \frac{(x-1)(x-2)}{1.2.3}(a-1)^3 + \&c.\right\}$$

$$= 1 + x\{A + Bx + Cx^2 + \&c.\} \text{ suppose,}$$

where A, (the sum of those terms within the bracket which are independent of x,) is easily found by putting $x = 0$ within the bracket, when there remains

$$(a-1) + \frac{-1}{2}(a-1)^2 + \frac{(-1)(-2)}{1.2.3}(a-1)^3 + \&c.,$$

$$\text{or } (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.,$$

which is therefore the value of A.

We have then $a^x = 1 + Ax + Bx^2 + Cx^3 + \&c.$, where A, B, C, &c. are functions of a, altogether independent of x, and will therefore remain the same for the same value of a, however we change that of x.

$$\text{Hence } a^{x+h} = 1 + A(x+h) + B(x+h)^2 + C(x+h)^3 + \&c.:$$

$$\text{but } a^{x+h} = a^x . a^h = \{1 + Ax + Bx^2 + Cx^3 + \&c.\}\, a^h;$$

therefore, equating coefficients of x in these identical values of a^{x+h}, we have

$$A + 2Bh + 3Ch^2 + \&c. = Aa^h = A\{1 + Ah + Bh^2 + Ch^3 + \&c.\}:$$

hence, equating coefficients of different powers of h in the above, we get

$$2B = A^2,\ 3C = AB, \&c., \text{ or } B = \frac{A^2}{1.2},\ C = \frac{AB}{3} = \frac{A^3}{1.2.3}, \&c.;$$

$$\text{so that we find} \quad a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.,$$

$$\text{where } A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

146. Since the above result is true for all values of x, take x such that $Ax = 1$; then

$$x = \frac{1}{A}, \text{ and } a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c. = 2.7182818 \; \&c.,$$

which number it is usual to denote by e; hence $a^{\frac{1}{A}} = e$, or $a = e^A$, and therefore $A = \log_e a$, and

$$a^x = 1 + (\log_e a)\,x + (\log_e a)^2\,\frac{x^2}{1.2} + (\log_e a)^3\,\frac{x^3}{1.2.3} + \&c.:$$

whence also, writing e for a, we get, since (131) $\log_e e = 1$,

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

147. We have

$$\left(1 + \frac{x}{n}\right)^n = 1 + n\,\frac{x}{n} + \frac{n(n-1)}{1.2}\,\frac{x^2}{n^2} + \&c. = 1 + x + \frac{n-1}{n}\,\frac{x^2}{1.2} + \&c.$$

$$= 1 + x + \left(1 - \frac{1}{n}\right)\frac{x^2}{2} + \left(1 - \frac{1}{n}\right)\left(\frac{1}{2} - \frac{1}{n}\right)\frac{x^3}{3} + \&c.$$

Now, as n increases, it is plain that $\frac{1}{n}$ decreases, and tends to zero as its *Limit:* hence we see that, as n increases, the Limit of $\left(1+\frac{x}{n}\right)^n$ is $1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+$ &c. or e^x.

148. By (146) we have $(e^x-1)^n=(x+\frac{1}{2}x^2+\&c.)^n=x^n+\&c.$ (i): but we have also $(e^x-1)^n=e^{nx}-n\,e^{(n-1)x}+\frac{1}{2}n(n-1)\,e^{(n-2)x}-\&c.$ (ii), in which each of the quantities e^{nx}, $e^{(n-1)x}$, &c. may be expanded by (146), so that the whole coefficient of x^r in (ii) will be

$$\frac{n^r}{1.2\ldots r}-n\frac{(n-1)^r}{1.2\ldots r}+\tfrac{1}{2}n\,(n-1)\frac{(n-2)^r}{1.2\ldots r}-\&c.:$$

but in (i) the coeff. of x^r is zero, if $r<n$, and it is 1, if $r=n$; hence, since the two expressions for $(e^x-1)^n$ must be identical,

$$n^r-n\,(n-1)^r+\tfrac{1}{2}n\,(n-1)\,(n-2)^r-\&c.=0,\ \text{if}\ r<n,$$

and $$n^n-n\,(n-1)^n+\tfrac{1}{2}n\,(n-1)\,(n-2)^n-\&c.=1.2.3\ldots n.$$

149. In the value of $\log_e a=A=(a-1)-\frac{1}{2}(a-1)^2+\&c.$,

write $1+x$ for a, and $\therefore x$ for $a-1$;

then $$\log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\&c.\ldots.(i)$$

Hence we might proceed to find the logarithms of numbers; thus

$$\log_e 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\&c.=\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\&c.:$$

but the series thus obtained are not sufficiently convergent, as it would take very many terms to obtain the logarithm accurately to six or seven decimal places.

150. A more convergent series, however, may be obtained as follows from the above.

Since $$\log_e(1+x)=\quad x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\&c.,$$

$$\therefore \log_e(1-x)=-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4-\&c.,$$

by writing $-x$ for x: and hence

$$\log_e(1+x)-\log_e(1-x)\ \text{or}\ \log_e\frac{1+x}{1-x}=2\left\{x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\&c.\right\}.$$

In this series write $\frac{m-n}{m+n}$ for x, and $\therefore \frac{m}{n}$ for $\frac{1+x}{1-x}$;

$$\therefore \log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c.\right\} \ldots (\alpha)$$

If $n = 1$, then

$$\log_e m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \frac{1}{5}\left(\frac{m-1}{m+1}\right)^5 + \&c.\right\} \ldots (\beta)$$

a convergent series, from which the logarithms of low primes may be found.

Thus $\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \&c.\right\} = .6931471806$, as may be seen by taking nine terms of the series.

151. Again, if in (α) we put $m = n + 1$, we get

$$\log_e \frac{n+1}{n} = \log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3}\frac{1}{(2n+1)^3} + \&c.\right\} (\gamma),$$

a very convergent series, by means of which, having given the logarithm of one of two consecutive numbers, n and $n + 1$, we may find that of the other.

Thus $\log_e 9 - \log_e 8 = 2\log_e 3 - 3\log_e 2 = 2\left\{\frac{1}{17} + \frac{1}{3.17^3} + \&c.\right\}$; and by means of *four* terms of this series, having given $\log_e 2$ above, we may find $\log_e 3 = 1.0986122884$.

And $\log_e 5 - \log_e 4 = \log_e 5 - 2\log_e 2 = 2\left\{\frac{1}{9} + \frac{1}{3.9^3} + \&c.\right\}$, by taking *five* terms of which series we get $\log_e 5 = 1.6094379122$, and, adding $\log_e 2$ to this, we have $\log_e 10 = 2.3025850928$, and thence, by common division, the modulus M

$$= 1 \div \log_e 10 = .4342944819.$$

152. Once more in (γ) write x^2 for $n + 1$, and, therefore, $2x^2 - 1$ for $2n + 1$; then

$$\log_e \frac{x^2}{x^2 - 1} = 2\log_e x - \log_e(x+1) - \log_e(x-1)$$

$$= 2\left\{\frac{1}{2x^2 - 1} + \frac{1}{3(2x^2-1)^3} + \&c.\right\} \ldots\ldots (\delta),$$

a very rapidly converging series, by means of w having given the logarithms of any two of three secutive numbers, $n+1$, n, $n-1$, we may find th the other.

Thus, taking the numbers 5, 6, 7, of which the logarithms and 6 (= 2 × 3) are now known, we have

$$2\log_e 6-\log_e 7-\log_e 5=2\left\{\frac{1}{71}+\frac{1}{3(71)^3}+\&c.\right\},$$

whence we may find $\log_e 7$, &c.

Or the same may be found by (γ) as follows:

$$\log_e 50-\log_e 49,\text{ or }(\log_e 2+2\log_e 5)-2\log_e 7=2\{\tfrac{1}{99}+\&c.\}.$$

153. By means of the above (or other similar formula a table of Napierian logarithms may be formed; an then these may be converted into logarithms to th base 10 or to any other base, by multiplying eac by the proper modulus.

Thus $\log_{10}2=M\log_e 2=.4342944819\times.6931471806=.301030$ and so we might find, if desired, $\log_3 2=(1\div\log_e 3)\times\log_e 2$, &c.

Or, having once obtained $\log_e 10$ and, by means of i the *modulus*, we may write at once in (α)

$$\log_{10}\frac{m}{n}=M\log_e\frac{m}{n}=2M\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\&c.\right\};$$

and so we get, corresponding to the formulæ (β), (γ), (δ above,

$$\log_{10}m=2M\left\{\frac{m-1}{m+1}+\&c.\right\},\ \log_{10}(n+1)-\log_{10}n=2M\left\{\frac{1}{2n+1}+\&\right.$$

$$\text{and }\ 2\log_{10}x-\log_{10}(x+1)-\log_{10}(x-1)=2M\left\{\frac{1}{2x^2-1}+\&c\right.$$

and thus may calculate immediately the common lo arithms, without finding the Napierian.

154. We are now able to prove the statement ma in (143).

For $\delta=\log_{10}(N+d)-\log_{10}N=\log_{10}\frac{N+d}{N}=M\log_e\left(1+\frac{d}{N}\right)$

$$=\text{by (i) } M\left\{\frac{d}{N}-\text{\&c.}\right\}=\frac{M}{N}d \text{ nearly,}$$

when d is small compared with N: that is, $\delta \propto d$, or the increase of the logarithm is proportional to the increase of the number.

Also $D=\log_{10}\frac{N+1}{N}=M\log_e\left(1+\frac{1}{N}\right)=\frac{M}{N}$ nearly;

now $M = .43$ &c, and is, therefore, $<\frac{1}{2}$, while N, if a number of five places, is > 10000; hence we have $D<\frac{1}{20000}<.00005$ in every case, and $\therefore \frac{D}{1000}$ would at the utmost only have its first significant figure in the *eighth* place of decimals, and so the term $\gamma\frac{D}{1000}$ in (144) could only, if at all, affect the *seventh* or last figure of the mantissa, but would not generally affect it at all: thus to 7 places of decimals the logs of 35714.23 and 35714.232198 &c. would probably coincide, the difference appearing in the 8th and following places. Hence with Tables, which give mantissæ only to 7 figures, we can only expect to find the numbers, which correspond to given logarithms, correct in their first seven figures.

155. As the sines and cosines of all angles, and the tangents of angles less than 45°, are all less than 1, their logarithms would be *negative*. To avoid the awkwardness of printing these, the logs of *all* the Ratios are increased by 10, and the true log of any Ratio is, therefore, to be found from the Tabular Log by diminishing it by 10: so that, using Log sin$A$, Log cos$A$, &c., or L sin$A$, *L cosA, &c.,* for the Tabular logarithms, we

have log-sin$A$ = L sin$A$ − 10, &c. Of course, as in (102), the same column, that serves, with a *descending* column of minutes, for Log-sin$A$, will serve also, with an *ascending* column of minutes, for Log-cos(90°−A), &c.; so that the Tables of Log-functions need not be carried beyond 45°.

Again, since $\sin A = \dfrac{1}{\operatorname{cosec} A}$, we have

$$\text{log-sin}A = \log 1 - \text{log-cosec}A = -\,\text{log-cosec}A;$$

whence $(\mathrm{L}\sin A - 10) = -(\mathrm{L}\operatorname{cosec} A - 10)$,

or $\mathrm{L}\sin A = 20 - \mathrm{L}\operatorname{cosec} A$, $\mathrm{L}\operatorname{cosec} A = 20 - \mathrm{L}\sin A$.

And similarly for the other pairs of reciprocal functions.

156. We shall now shew that the principle of (104) applies also to the *logarithmic*, as well as to the natural functions, of angles, namely, that for a small increment of the angle the increment of the log-sine, &c. is proportional to that of the angle, with the same cases of exception as before.

For let f denote the value of any of the *natural* functions of θ; then, by (104), $f+k\delta$ will be the approximate value of the same function of $\theta+\delta$, where k is some constant quantity depending only upon θ, not upon δ, [thus for the *sine*, k is $\cos\theta$, for the cosine, k is $-\sin\theta$, &c., as appears in (104),] δ being supposed very small, and θ itself not in the excepted cases.

Now $\log(f+k\delta) - \log f = \log\dfrac{f+k\delta}{f} = \log\left(1+\dfrac{k}{f}\delta\right)$

$$= M\left(\frac{k}{f}\delta - \frac{1}{2}\frac{k^2}{f^2}\delta^2 + \&\text{c.}\right) = M\frac{k}{f}\delta, \text{nearly, where } M = \frac{1}{\log_e 10}.$$

So that, whatever be the function in question, we shall have the increment of the log-function, and, therefore, also of the Log-function, of any angle proportional to the small increment of the angle, provided that the angle itself be not in one of the excepted cases, namely, near 90° for the sine, tangent, and secant, (that is, for those functions which *increase* from 0° to 90°,) and near 0° or 180° for the cosine, cotangent, and cosecant, (that is, for those functions which *decrease* from 0° to 90°).

Hence we may apply precisely the same method as in (103) to find, approximately, the Log-function of any angle not exactly given in the Tables, or, conversely, to find the angle from its Log-function.

157. Moreover, since

$$\text{L}\operatorname{cosec}(A+n'')-\text{L}\operatorname{cosec}A=\text{log-cosec}(A+n'')-\text{log-cosec}A$$
$$=-\text{log-sin}(A+n'')+\text{log-sin}A=-\{\text{L}\sin(A+n'')-\text{L}\sin A\},$$

it appears that the *differences* for the Log-cosecants will be the same as for the Log-sines, being *increments* for the one and *decrements* for the other. Accordingly, in any set of Tables, the successive values of D for the sine and cosecant, for the intervals at which those Tables are constructed, will be found registered in the same vertical column. Similarly for the cosine and secant, tangent and cotangent.

158. If in $\big(104\,(1)\big)$ we put δ for $\sin\delta$, and $\frac{1}{2}\delta$ for $\tan\frac{1}{2}\delta$, we get

$$\sin(\theta+\delta)-\sin\theta=\delta\cos\theta(1-\tfrac{1}{2}\delta\tan\theta)=\delta\cos\theta-\tfrac{1}{2}\delta^2\sin\theta,$$

$$\text{or}\quad\frac{\sin(\theta+\delta)}{\sin\theta}=1+(\delta\cot\theta-\tfrac{1}{2}\delta^2),$$

$$\text{and}\quad\text{L-}\sin(\theta+\delta)-\text{L}\sin\theta=\text{log-sin}(\theta+\delta)-\text{log-sin}\,\theta$$
$$=\log\frac{\sin(\theta+\delta)}{\sin\theta}=M\{(\delta\cot\theta-\tfrac{1}{2}\delta^2)-\tfrac{1}{2}(\delta\cot\theta-\tfrac{1}{2}\delta^2)^2+\&\text{c.}\}$$
$$=M\delta\cot\theta-\tfrac{1}{2}M\delta^2(1+\cot^2\theta),\ \text{nearly.}$$

Consequently, if $\cot\theta$ be very large, or θ near 0°, we cannot omit the term involving δ^2, that is, we cannot assume *the principle* that the increment of the Log-sine is proportional to that of the angle: and the same may be shewn to be true for the tangent. Now it very frequently happens in practice that a very small angle has to be determined from its sine or tangent. Accordingly in the best sets of Tables, a table is given of Log-sines and Log-tangents for every second in the first two degrees.

159. If we have to determine accurately an angle, which is given by some one of its functions in one of the excepted cases, we must replace the given equation by some other not liable to the same objection.

Thus, if we have given $\sin A = a$, where $a = 1$ nearly, or A is near 90°, we may write

$$\sin(45^\circ - \tfrac{1}{2}A) = \sqrt{[\tfrac{1}{2}\{1-\cos(90^\circ - A)\}]} = \sqrt{\{\tfrac{1}{2}(1-\sin A)\}} = \sqrt{\{\tfrac{1}{2}(1-a)\}},$$

where the angle $45^\circ - \frac{1}{2}A$ will be near 0°, and can be computed accurately from its sine.

So if we have given $\cos A = a$, where $a = 1$ nearly, or A is small, we may write

$$\sin\tfrac{1}{2}A = \sqrt{\{\tfrac{1}{2}(1-\cos A)\}} = \sqrt{\{\tfrac{1}{2}(1-a)\}},$$ where $\frac{1}{2}A$ is near 0°:

or, if $\tan A = a$, where a is very great, or A is near 90°, we may write

$$\tan(A - 45^\circ) = \frac{\tan A - 1}{\tan A + 1} = \frac{a-1}{a+1},$$ where $A - 45^\circ$ is near 45°.

Ex. 28.

1. Given $\text{L}\sin 11^\circ 24' = 9.2959129$, and $\text{L}\sin 11^\circ 25' = 9.2965390$; find $\text{L}\sin 11^\circ 24' 25''$, and $\text{Log-sin}^{-1} 9.2959875$.
2. Given $\text{L}\cos 60^\circ 1' = 9.6987511$, find $\text{L}\cos 60^\circ 0' 20''$ and $\text{L}\sin 29^\circ 59' 15''$.
3. Given $\text{L}\tan 44^\circ 59' = 9.9997473$, find $\text{L}\cot 44^\circ 59' 20''$.
4. Given $\text{L}\operatorname{cosec} 30^\circ 1' = 10.3008113$, find $\text{L}\sin 30^\circ 0' 45''$ and $\text{Log-sin}^{-1} 9.6990999$.
5. Shew that $\text{L}\sin 2A = \text{L}\sin A + \text{L}\cos A + \log 2 - 10$; and find the values of $\text{L}\sin A$ and $\text{L}\cos A$, when $A = 30^\circ$.
6. Shew that $\text{L}\operatorname{vers} A = \log 2 + 2\text{L}\sin\frac{1}{2}A - 10$, and, by means of this formula, find $\text{L}\operatorname{vers} 60^\circ$. How may it be found more simply?

ANSWERS TO THE EXAMPLES.

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1. 1. 74° 22′ 58″; −52° 15′ 29″; 88° 58′ 56″.8.
2. 34° 44′ 40″; 207° 17′ 7″.
3. 80° 50′ 51″; 9°.1525, 80°.8475. 4. 57° 12′, 32° 48′.
5. 60° 0′ 2″. 6. 72°, 54°, 54°. 7. 70°, 50°, 60°.
8. 80°, 40°, 60°. 9. 30°, 60°, 90°. 10. $81\frac{6}{11}°$, $40\frac{10}{11}°$, $57\frac{6}{11}°$.

---

2. 1. 115^g; $72^g\,75^{`}$; $99^g\,67^{`}\,99^{``}$.
2. $198^g\,97^{`}\,97^{``}$; $110^g\,99^{`}\,96^{``}$; $-25^g\,10^{`}\,1^{``}$.
3. 33° 26′ 8″, 56° 33′ 52″, $62^g\,84^{`}\,93^{``}$.
4. $1^g\,11^{`}\,11^{``}$, $3^{``}.08$; 54′, 32″.4.
5. 108°, 120°. 7. 1 : 3. 8. 54°, 81°, 108°, 135°, 162°.
10. 7 and 5. 11. 144°, 108°. 12. 16 or 9.

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3. 1. .157 *in.* 2. 68° 45′ 18″. 3. $5\frac{1}{3}$, $\frac{2}{9}\pi$. 4. 5 *ft* 9*in.*
5. 25.77 *sq. ft*; 773 *cub. ft.* 6. 2.57 *sq. ft.* 7. 39.24 *yds.*
8. 9 : 16. 9. 1 : 15 : 360. 10. 1148 *miles.*
11. 69.8 *miles*; 25143 *miles.* 12. $17\frac{1}{2}$ *miles.*

---

4. 1. $\frac{2}{5}\pi$, $\frac{1}{10}\pi$, $\frac{2}{5}\pi$. 2. $\frac{101}{8000}\pi$. 3. $\frac{1}{4}\pi$, $\frac{1}{3}\pi$, $\frac{5}{12}\pi$.
4. $\frac{1}{3}$; 19° 5′ 55″. 5. 5° 43′ 46″. 6. $\frac{7}{640}\pi$. 7. 100 : 81.
8. $88\frac{2}{7}°$; $\frac{304}{315}\pi$. 9. 4; 214° 51′ 33″, 229° 10′ 59″.
10. $\frac{3}{4}$. 11. $r'\left(\frac{\pi}{2}+\frac{a}{r}\right)$. 12. 72° 53′ 33″.

---

5. 1. $\frac{\sqrt{(\sec^2-1)}}{\sec}$, $\frac{1}{\sqrt{(1+\cot^2)}}$. 2. $\frac{\sqrt{(\operatorname{cosec}^2-1)}}{\operatorname{cosec}}$, $\frac{\cot}{\sqrt{(1+\cot^2)}}$.
3. $\frac{\sin}{\sqrt{(1-\sin^2)}}$, $\frac{1}{\sqrt{(\operatorname{cosec}^2-1)}}$. 4. $\frac{\cos}{\sqrt{(1-\cos^2)}}$, $\frac{1}{\sqrt{(\sec^2-1)}}$.
5. $\frac{\sqrt{(1+\cot^2)}}{\cot}$, $\frac{\sec}{\sqrt{(\sec^2-1)}}$. 6. $\sqrt{(2\,\text{vers}-\text{vers}^2)}$.
7. $\frac{3}{5}$, $\frac{4}{5}$, $1\frac{1}{3}$, $1\frac{1}{4}$, $1\frac{2}{3}$. 8. $\frac{1}{2}\sqrt{5}$, $\frac{2}{5}\sqrt{5}$.
9. $\sqrt{5}$, $\frac{2}{5}\sqrt{5}$. 10. $\frac{1}{4}\sqrt{15}$, $\frac{3}{4}$.

---

6. 1. $31^\circ\, 33'\, 10''$. 3. $73:64$. 4. $2:1$; $76^\circ\, 23'\, 40''$, $42^\circ\, 44'\, 13''$.

6. ± 1. 7. $\frac{9}{\sqrt{85}}$, $\frac{2}{\sqrt{85}}$, $4\frac{1}{2}$. 8. $\frac{1}{\sqrt{2}}$ or $\frac{3}{\sqrt{10}}$; 1 or 3.

10. $\frac{1}{2}$ or $\frac{2}{3}$. 11. $\frac{a^2+b^2}{a^2-b^2}$; $\frac{ab}{a^2+b^2}$. 12. $A = \pm 45^\circ$, $B = \pm 30^\circ$.

---

7. 1. $\operatorname{Sin} A = \frac{r \operatorname{Cos} A}{\sqrt{(\operatorname{Cosec}^2 A - r^2)}}$.

2. $\operatorname{Vers} A = \frac{r \operatorname{Sec} A - r^2}{\operatorname{Sec} A} = \frac{r \operatorname{Cosec} A - r \operatorname{Cot} A}{\operatorname{Cosec} A}$.

3. $\operatorname{Sin} A \operatorname{Cos} A = \frac{r^2}{\operatorname{Tan} A + \operatorname{Cot} A}$.

4. $\operatorname{Cot}^2 A \operatorname{Cos}^2 A = r^2 \operatorname{Cot}^2 A - r^2 \operatorname{Cos}^2 A$.

5. $\operatorname{Sec}^2 A \operatorname{Cosec}^2 A = r^2 \operatorname{Sec}^2 A + r^2 \operatorname{Cosec}^2 A$.

6. $\operatorname{Tan} A + \operatorname{Cot} A = \sqrt{(\operatorname{Sec}^2 A + \operatorname{Cosec}^2 A)}$.

7. $\operatorname{Sin} A = \frac{r^2}{\operatorname{Cosec} A}$. 8. $\operatorname{Tan} A \operatorname{Cot} A = r^2$.

9. $\operatorname{Tan} A = \frac{r \operatorname{Sin} A}{\cos A}$. 10. $r \operatorname{Sin} A = 2 \operatorname{Sin} \frac{1}{2} A \operatorname{Cos} \frac{1}{2} A$.

11. $r \operatorname{Cos} A = r^2 - 2 \operatorname{Sin}^2 \frac{1}{2} A$. 12. $\operatorname{Chd}^2 A = 2r \operatorname{Vers} A$.

---

8. 1. $-\frac{1}{2}\sqrt{3}$, 2; $\frac{1}{2}\sqrt{2}$, -1; $-\sqrt{3}$, -2. 2. $-\frac{1}{2}\sqrt{3}$, $-\frac{1}{2}$, $\sqrt{3}$.

3. $\sqrt{3}$, $1\frac{1}{2}$; 1, $\frac{1}{2}$. 4. $1\frac{1}{2}$; $\frac{1}{2}(2+\sqrt{2})$.

5. $+\cos 10^\circ$, $-\cos 20^\circ$, $-\cot 30^\circ$, $+\cot 40^\circ$, $-\sec 40^\circ$, $-\sec 30^\circ$.

6. $2\cos^2\theta$. 7. 1. 8. 0. 9. $+\cos\theta$; $\mp\cot\theta$. 10. $\pm\operatorname{cosec} a$.

---

9. 1. $(4n+1)\frac{1}{4}\pi$. 2. $(4n\pm 1)\frac{1}{8}\pi$. 3. $(6n\pm 2)\frac{1}{6}\pi$.

4. $(6n+1)\frac{1}{6}\pi$. 5. $(3n+1)\frac{1}{15}\pi$. 8. 26 *ft.*; 15 *ft.*

9. 156 *ft.* 10. 51 *ft.* 11. 79 *ft.* 12. 36 *ft.*; 63 *ft.*

---

10. 5. $2\sin A\cos A$, $2\cos^2 A - 1$. 6. $\frac{\cot A\cot B \mp 1}{\cot B \pm \cot A}$.

7. $\frac{\sec A \sec B}{1 \mp \sqrt{\{(\sec^2 A - 1)(\sec^2 B - 1)\}}}$.

8. $\frac{\operatorname{cosec} A \operatorname{cosec} B}{\sqrt{(\operatorname{cosec}^2 B - 1)} \pm \sqrt{(\operatorname{cosec}^2 A - 1)}}$.

---

13. 1. $\frac{1}{2}(\cos 4A - \cos 6A)$, $\frac{1}{2}(\cos A - \cos 5A)$, $\frac{1}{2}(\cos \frac{1}{2}A - \cos \frac{5}{2}A)$
$\sin^2 3A - \sin^2 2A$, $\sin^2 \frac{5}{2}A - \sin^2 \frac{1}{2}A$, $\sin^2 \frac{5}{4}A - \sin^2 \frac{1}{4}A$.

2. $\frac{1}{2}(\cos \frac{7}{2}A + \cos \frac{9}{2}A)$, $\frac{1}{2}(\cos 3A + \cos A)$, $\frac{1}{2}(\cos \frac{5}{4}A + \cos \frac{3}{4}A)$
$\cos^2 \frac{7}{4}A - \sin^2 \frac{9}{4}A$, $\cos^2 \frac{3}{2}A - \sin^2 \frac{1}{2}A$, $\cos^2 \frac{5}{8}A - \sin^2 \frac{3}{8}A$.

3. $\frac{1}{2}(\sin 5A - \sin A)$, $\frac{1}{2}(\sin 3A - \sin 2A)$, $\frac{1}{2}(\sin\frac{5}{2}A - \sin\frac{1}{2}A)$;
$\cos^2(45^\circ + \frac{1}{2}A) - \sin^2(45^\circ - \frac{5}{2}A)$, $\cos^2(45^\circ + A) - \sin^2(45^\circ - \frac{3}{2}A)$,
$\sin^2(45^\circ - \frac{1}{4}A) - \sin^2(45^\circ - \frac{5}{4}A)$.
4. $\frac{1}{2}\{\cos(A + B - C) - \cos(A + B + C)\}$,
$\frac{1}{2}\{\cos(A - B + C) + \cos(A - B - C)\}$, $\frac{1}{2}(\sin 2A + \sin 2B)$;
$\sin^2\frac{1}{2}(A + B + C) - \sin^2\frac{1}{2}(A + B - C)$,
$\cos^2\frac{1}{2}(A - B + C) - \sin^2\frac{1}{2}(A - B - C)$, $\cos^2(45^\circ - B) - \sin^2(45^\circ - A)$.

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14. 1. $2\sin 4A \cos A$; $-2\sin\frac{3}{4}A \sin\frac{1}{4}A$.
2. $2\sin(A - B)\cos B$; $2\sin(45^\circ - A + \frac{1}{2}B)\cos(45^\circ - \frac{1}{2}B)$.
3. $2\sin\frac{1}{2}A\cos(\frac{1}{2}A + B)$; $4\sin\frac{3}{8}A\cos\frac{1}{8}A$.
4. $2\sin\frac{3}{2}A\sin\frac{1}{2}A$; $2\cos\frac{1}{2}(A - C)\cos\frac{1}{2}(A - 2B + C)$.
5. $\sin 2(A - B)\sin 2B$; $-\sin\frac{3}{2}A\sin\frac{1}{2}A$.
6. $-2\sin\frac{1}{2}(A + 3B)\sin\frac{1}{2}(3A - B)$; $2\sin\frac{3}{2}A\cos\frac{1}{2}A$.
7. $2\sin\frac{1}{2}(A + B + 2C)\sin\frac{1}{2}(A - B)$; $\sin(A - B)\sin(A - 3B)$.
8. $4\sin(A - \frac{1}{2}B)\sin\frac{1}{2}B$; $2\cos 45^\circ\cos(45^\circ - A)$ or $\sqrt{2}\cos(45^\circ - A)$.
9. $-\cos 2(A + B)$; $\cos 2A \cos 2B$.
10. $2\cos(45^\circ - B)\cos(45^\circ - A)$;
$-2\sin 45^\circ \sin(45^\circ - \frac{1}{2}A)$, or $-\sqrt{2}\sin(45^\circ - \frac{1}{2}A)$.

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16. 9. $\frac{1}{2}\sqrt{(2 + \sqrt{2})}$. 10. $\frac{1}{2}\sqrt{(2 - \sqrt{2})}$. 11. $\sqrt{\frac{2\sqrt{2} - \sqrt{3} + 1}{2\sqrt{2} + \sqrt{3} - 1}}$.
12. $\frac{\sqrt{(2\sqrt{2} - \sqrt{3} - 1)}}{2\sqrt[4]{2}}$. 13. $\frac{2\sqrt{2} - \sqrt{3} - 1}{2\sqrt{2}}$.
14. $\frac{1}{2}\sqrt{\{2 + \sqrt{(2 + \sqrt{2})}\}}$. 15. $\frac{1}{2}\sqrt{\{2 + \sqrt{(2 - \sqrt{2})}\}}$. 16. $\sqrt{(2 + \sqrt{2})}$.

---

7. 1. 9 : 14. 3. $2182\frac{1}{2}$ *miles.* 6. 141 *miles.* 7. 205 *miles.*
8. 4000 *miles.* 9. 8448 *miles.* 10. 8 *miles*, 13 *miles.*

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8. 1. .4166114; 24° 37′ 19″. 2. .3712598; 68° 12′ 37″.
3. 1.0003394; 45° 0′ 24″. 4. 1.7317213; 30° 0′ 43″.
5. 1.9994125; 59° 59′ 4″. 6. 1.1545810; 60° 0′ 30″.
12. 36° or 108°; or, in general terms, $(10n \pm 1)\frac{1}{5}\pi$, $(10n \pm 3)\frac{1}{5}\pi$.

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0. 1. $2\frac{1}{2}$ *sq. yds.* 2. 1559 *sq. yds.* 3. 415 *sq. yds.* 4. 750.
5. 1A 2R 17P $1\frac{3}{4}$ *sq. yds.* 6. 57 *yds.* 7. $90\frac{1}{9}$ *sq. yds.*
8. £101 15*s.* 9. $B = 60^\circ$; $a = 2.625$ *ft*, $b = 4.5465$ *ft*;
10. 2887 *sq. yds.* area = 6 *sq. ft.*
11. $B = 60^\circ$, $C = 90^\circ$; $b = 17$. 12. 90°; $\frac{3}{5}$, $\frac{4}{5}$.

24. 1. $R=\frac{1}{3}a\sqrt{3}$; $r=\frac{1}{6}a\sqrt{3}$. 2. $R=\dfrac{a^2}{\sqrt{(4a^2-c^2)}}$; $r=\dfrac{c}{2}\sqrt{\dfrac{2a-c}{2a+c}}$.

4. $AC=\sqrt{\dfrac{(ac+bd)\,(ad+bc)}{ab+cd}}$; $BD=\sqrt{\dfrac{(ac+bd)\,(ab+cd)}{ad+bc}}$.

7. $r=\dfrac{r_1r_2r_3}{r_1r_2+r_1r_3+r_2r_3}$. 12. 12; $1\frac{5}{7}$, $2\frac{1}{2}$.

8. $r=4R\sin\frac{1}{2}A\,\sin\frac{1}{2}B\,\sin\frac{1}{2}C$; $r_1=4R\sin\frac{1}{2}A\,\cos\frac{1}{2}B\,\cos\frac{1}{2}C$.

---

25. 1. 2A 2R 20P. 2. 94 *ft.* 3. $1426\frac{1}{2}$ *ft.* 4. 87 *ft.*
5. 2A 1R 9P. 6. 324 *yds.* 7. 513 *yds.* 8. 72 *ft.*
9. 3750 *sq. yds.* 10. 1099 *yds.*
11. 173 *yds,* 100 *yds,* 200 *yds*; 1A 2R 13P. 12. 150 *ft.*
13. 185 *yds.* 14. 30 *ft.* 15. 6 *miles,* 16 *miles.*
16. 14A 2R 27P $3\frac{1}{4}$ *sq. yds.* 17. 4 *miles.*
18. 4 *sq. yds* 8 *ft* $11\frac{1}{4}$ *in.* 19. $a\sin\alpha$, $a\cos\alpha$. 20. 101 *ft.*
21. 1442 *ft.* 22. 6 *miles.* 23. 243 *yds.* 24. 13° 53′ 52″.

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26. 1. .9030900, .9542426, 1.0791813.
2. 1.3010300, 1.3979400, 1.7781513.
3. $\bar{1}.5228787$, $\bar{1}.3979400$, $\bar{1}.6020600$.
4. $\bar{2}.4771213$, $\bar{2}.5228787$, $\bar{3}.5185140$.
5. .2552726, 2.1461280, .1583626.
6. $\bar{2}.7958800$, $\bar{4}.9172146$, .0211893.
7. 1.0253059, .6232493, .6434527.
8. $\bar{2}.0453230$, .0880456, .0416462.
9. $\bar{1}.5000000$, $\bar{1}.4336766$, .0527875.
10. $\bar{3}.4647060$, $\bar{1}.8957576$, $\bar{1}.5355396$.
11. $\bar{1}.8035760$. 12. $\bar{1}.8149306$.

---

27. 1. 1.5527807, $\bar{1}.5530013$, $\bar{3}.5528198$, .5530092.
2. 3571.693, .3573465, .00035709, .3572892. 3. 1.3010502, .0200087.
4. 6.4913649, 4.4913652, 1.4913722.
5. 4.498527. 6. .5738614.

Table for Ex. 3.

D.	Pro.	
217		217
	1	22
	2	43
	3	65
	4	87
	5	109
	6	130
	7	152
	8	174
	9	195

---

28. 1. 9.2961738; 11° 24′ 7″. 2. 9.6986781, 9.6985869.
3. 10.0001685. 4. 9.6991340; 30° 0′ 36″.
5. 9.6989700; 9.9375306. 6. 9.6989700.

Metcalfe and Palmer, Printers, Cambridge.

PLANE TRIGONOMETRY

PART II.

BY THE REV. J. W. COLENSO, M.A.
RECTOR OF FORNCETT ST. MARY, NORFOLK,
LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

LONGMAN AND CO., LONDON.

MDCCCLI.

*** The brackets ( ) are employed to refer to Articles in Part I,
[ ] to those in Part II.

In the Answers, the letter *u* is sometimes used, for shortness, to represent the quantity that may be given in the Text in any problem.

For the Miscellaneous Problems some of the following Logarithms will be required:

Logarithms, to base 10, *of all Prime Numbers from* 1 *to* 100.

No.	Logs.	No.	Logs.	No.	Logs.	No.	Logs.
2	0.3010300	19	1.2787536	43	1.6334685	71	1.8512583
3	0.4771213	23	1.3617278	47	1.6720979	73	1.8633229
7	0.8450980	29	1.4623980	53	1.7242759	79	1.8976271
11	1.0413927	31	1.4913617	59	1.7708520	83	1.9190781
13	1.1139434	37	1.5682017	61	1.7853298	89	1.9493900
17	1.2304489	41	1.6127839	67	1.8260748	97	1.9867717

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ERRATA: PART I. *First Edition.*

PAGE
67, line 11, for $= 0$ read $= -1$.
68, The words 'For all values of A, B, C, shew that' should *follow*, not *precede*, Ex. **15**. 10.
73, line 6, for *distance* read *height*.
83, line 9, for $(1+\cos\delta)$ read $(1-\cos\delta)$.
– line 10, for $\cot\frac{1}{2}\delta$ read $\tan\frac{1}{2}\delta$.
– line 28, in num^r^, for $-\sin\delta$ read $-\tan\delta$.
– line 33, in den^r^, for $\tan\frac{1}{2}\delta$ read $\tan\delta$.
96, Ex. **21**. 6, for S (four times) read s.
111, line 4, for 0 *and* $+\infty$ read $-\infty$ *and* 0,
– line 5, for 0 *and* $-\infty$ read 0 and $+\infty$.

ANSWERS.

Ex. **20**. 5, should be 3A. 2R. 18P. 11½ *sq. yds.*
Ex. **20**. 10, should be 1A. 3R. 6P. 8½ *sq. yds.*
Ex. **25**. 4, should be 260 *ft.*
Ex. **25**. 11, should be 1A. 3R. 6P., *nearly.*
Ex. **25**. 18, should be 23P. 9 *sq. yds.* 4½ *ft.*
Ex. **25**. 21, should be 721 *ft.*
Ex. **28**. 2, should be 9.6988970, 9.9888059.

ERRATA: PART II.

PAGE
10, line 11, for $\cos(n-2)\theta$ read $n\cos(n-2)\theta$.
11, last line, interchange denominators n and 2.
50, Ex. 219, for 25 : 27 read 4 : 3.
51, Ex. 233, for *height of C* read *height of B*.
56, Ex. 309, for $\tan\beta$, $\cot\frac{1}{2}\alpha$, $\tan\beta$, read $\cot\beta$, $\tan\frac{1}{2}\alpha$, $\cot\beta$.
60, Ex. 356, for π read 180.

PLANE TRIGONOMETRY

PART II

CHAPTER I.

ON THE SERIES REQUIRED FOR THE CALCULATION OF π.

THE special object of this Chapter is to shew the different steps, by which we arrive at the value of the number π, which is of such great importance in Trigonometry, and indeed in every branch of Mathematics. But, in the course of our reasoning, many interesting results and remarkable formulæ will arise.

1. *To shew that*

$$(\cos\alpha + \sqrt{-1}\ \sin\alpha)(\cos\beta + \sqrt{-1}\ \sin\beta)\ldots(\cos\lambda + \sqrt{-1}\ \sin\lambda)$$
$$= \cos(\alpha + \beta + \ldots + \lambda) + \sqrt{-1}\ \sin(\alpha + \beta + \ldots + \lambda).$$

We have by Mult^n $(\cos\alpha + \sqrt{-1}\ \sin\alpha)(\cos\beta + \sqrt{-1}\ \sin\beta)$

$$= (\cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta) + \sqrt{-1}\,(\sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta)$$
$$= \cos(\alpha + \beta) + \sqrt{-1}\ \sin(\alpha + \beta).$$

Suppose this Law true for any number of such factors, so that

$$(\cos\alpha + \sqrt{-1}\ \sin\alpha)(\cos\beta + \sqrt{-1}\ \sin\beta)\ \ldots\ (\cos\kappa + \sqrt{-1}\ \sin\kappa)$$
$$= \cos(\alpha + \beta + \ldots\ \kappa) + \sqrt{-1}\ \sin(\alpha + \beta + \ldots\ \kappa)$$
$$= \cos\varsigma + \sqrt{-1}\ \sin\varsigma,\ \text{suppose};$$

then, introducing another factor, $\cos\lambda + \sqrt{-1}\ \sin\lambda$, we get

$$(\cos\alpha + \sqrt{-1}\ \sin\alpha)\ldots(\cos\kappa + \sqrt{-1}\ \sin\kappa)(\cos\lambda + \sqrt{-1}\ \sin\lambda)$$
$$= (\cos\varsigma + \sqrt{-1}\ \sin\varsigma)(\cos\lambda + \sqrt{-1}\ \sin\lambda)$$
$$= \cos(\varsigma + \lambda) + \sqrt{-1}\ \sin(\varsigma + \lambda)\ \text{by the previous result,}$$
$$= \cos(\alpha + \beta + \ldots + \kappa + \lambda) + \sqrt{-1}\ \sin(\alpha + \beta + \ldots + \kappa + \lambda).$$

Hence, since the Law has been shewn to be true for *two* factors, it is also true for *three*, and, therefore, for *four*, and so on, by Induction, *for* any number of such factors.

2. *To express* $\tan(\alpha+\beta+\gamma+\&c.)$ *in terms of* $\tan\alpha$, $\tan\beta$, $\tan\gamma$, &c.

From the result of [1] we obtain

$$\cos(\alpha+\beta+\gamma+\&c)+\sqrt{-1}\,\sin(\alpha+\beta+\gamma+\&c)$$
$$=(\cos\alpha+\sqrt{-1}\,\sin\alpha)(\cos\beta+\sqrt{-1}\,\sin\beta)(\cos\gamma+\sqrt{-1}\,\sin\gamma)\dots$$
$$=\cos\alpha(1+\sqrt{-1}\,\tan\alpha)\cos\beta(1+\sqrt{-1}\,\tan\beta)\cos\gamma(1+\sqrt{-1}\,\tan\gamma)\dots$$
$$=\cos\alpha\cos\beta\cos\gamma\dots(1+\sqrt{-1}\,\tan\alpha)(1+\sqrt{-1}\,\tan\beta)(1+\sqrt{-1}\,\tan\gamma)\dots$$
$$=\cos\alpha\cos\beta\cos\gamma\dots\{1+\sqrt{-1}\,S_1-S_2-\sqrt{-1}\,S_3+S_4+\&c\},\text{ by Mult}^{\text{n}},$$

where $S_1=\tan\alpha+\tan\beta+\tan\gamma+\dots$

$S_2=\tan\alpha\tan\beta+\tan\alpha\tan\gamma+\tan\beta\tan\gamma+\dots$

$S_3=\tan\alpha\tan\beta\tan\gamma+\&c$, &c = &c.

Therefore, equating possible and impossible parts of this expression, we have

$$\cos(\alpha+\beta+\gamma+\&c)=\cos\alpha\cos\beta\cos\gamma\dots(1-S_2+S_4-\&c),$$
$$\sin(\alpha+\beta+\gamma+\&c)=\cos\alpha\cos\beta\cos\gamma\dots(S_1-S_3+S_5-\&c),$$

and, therefore, $\tan(\alpha+\beta+\gamma+\&c)=\dfrac{S_1-S_3+S_5-\&c}{1-S_2+S_4-\&c}$.

Let the number of angles be n: then, if n be odd, the last term of the num$^{\text{r}}$ will be $(-1)^{\frac{1}{2}(n-1)}S_n$, and of the den$^{\text{r}}$ $(-1)^{\frac{1}{2}(n-1)}S_{n-1}$; or, if n be even, they will be $(-1)^{\frac{1}{2}(n-2)}S_{n-1}$, $(-1)^{\frac{1}{2}n}S_n$, respectively.

If in the above we suppose α, β, &c, each equal to θ, we may get from the preceding results the values of $\sin n\theta$, $\cos n\theta$, $\tan n\theta$, when n is a positive integer, in terms of θ. But this may be better obtained by proving first the following Theorem.

3. DEMOIVRE'S THEOREM. *To shew that for all values of* m, $(\cos\theta\pm\sqrt{-1}\,\sin\theta)^m=\cos m\theta\pm\sqrt{-1}\,\sin m\theta$.

We need not trouble ourselves with the double sign; for, if we can shew that $(\cos\theta+\sqrt{-1}\,\sin\theta)^m=\cos m\theta+\sqrt{-1}\,\sin m\theta$, then, writing $-\theta$ for θ, we get at once

$$(\cos\theta-\sqrt{-1}\,\sin\theta)^m=\cos m\theta-\sqrt{-1}\,\sin m\theta.$$

First, let the index be a *positive integer.*

In the result of [1] let there be m such factors, each of them being $\cos\theta+\sqrt{-1}\,\sin\theta$:

then $(\cos\theta+\sqrt{-1}\,\sin\theta)^m=\cos(\theta+\theta+\&c)+\sqrt{-1}\,\sin(\theta+\theta+\&c)$

$$=\cos m\theta+\sqrt{-1}\,\sin m\theta.$$

Next, let the index be a *negative integer.*

Then $(\cos\theta+\sqrt{-1}\,\sin\theta)^{-m}$

$= \dfrac{1}{(\cos\theta+\sqrt{-1}\,\sin\theta)^m} = \dfrac{1}{\cos m\theta+\sqrt{-1}\,\sin m\theta}$, by the former case,

$= \dfrac{\cos^2 m\theta+\sin^2 m\theta}{\cos m\theta+\sqrt{-1}\,\sin m\theta} = \cos m\theta - \sqrt{-1}\,\sin m\theta$, by common Div^n,

$= \cos(-m\theta)+\sqrt{-1}\,\sin(-m\theta).$

Lastly, let the index be a *fraction*, positive or negative.

Then, since

$\left(\cos\dfrac{m}{n}\theta+\sqrt{-1}\,\sin\dfrac{m}{n}\theta\right)^n = \cos m\theta+\sqrt{-1}\,\sin m\theta = (\cos\theta+\sqrt{-1}\,\sin\theta)^m$,

therefore, taking the n^{th} root on both sides, we have

$$\{\cos\theta+\sqrt{-1}\,\sin\theta\}^{\frac{m}{n}} = \cos\frac{m}{n}\theta + \sqrt{-1}\,\sin\frac{m}{n}\theta.$$

In this last case, however, it should be noticed that the quantity $(\cos\theta+\sqrt{-1}\,\sin\theta)^{\frac{m}{n}}$, in which an n^{th} root is to be taken, has (by the principles of Algebra) n distinct values, and that $\cos\dfrac{m}{n}\theta+\sqrt{-1}\,\sin\dfrac{m}{n}\theta$ is only one of these.

Hence then, for all values of m, positive or negative, integral or fractional, we have, as above stated,

$$(\cos\theta\pm\sqrt{-1}\,\sin\theta)^m = \cos m\theta\pm\sqrt{-1}\,\sin m\theta.$$

4. We may thus find the n values, above-mentioned, of

$$(\cos\theta+\sqrt{-1}\,\sin\theta)^{\frac{m}{n}}.$$

Since $\cos\theta=\cos(2r\pi+\theta)$, and $\sin\theta=\sin(2r\pi+\theta)$,

$$\therefore\ (\cos\theta+\sqrt{-1}\,\sin\theta)^{\frac{m}{n}} = \{\cos(2r\pi+\theta)+\sqrt{-1}\,\sin(2r\pi+\theta)\}^{\frac{m}{n}}$$
$$= \cos\frac{m}{n}(2r\pi+\theta)+\sqrt{-1}\,\sin\frac{m}{n}(2r\pi+\theta);$$

which last expression, by writing 0, 1, 2, &c. $(n-1)$, for r, will have n different values, and no more.

For (i), if we write 0, 1, 2, &c $(n-1)$, for r, we get n different values of it; since, if any two were identical, as, for instance, when $r=p$ and when $r=q$, (where p and q are each less than n,) then the angles $\dfrac{m}{n}(2p\pi+\theta)$ and $\dfrac{m}{n}(2q\pi+\theta)$ must differ by a multiple of 2π, or $\dfrac{m}{n}(p-q)$ must be an integer. But this is impossible, since m is prime *to* n, and p, q, are each less than n.

And (ii), if we take for r any value beyond the limits 0 and $(n-1)$, we shall get no new value; as, for instance, if $r = kn + r'$, (where r' is positive and less than n, and k may be any integer, positive or negative,) then

$$\cos\frac{m}{n}\{2(kn+r')\pi+\theta\}+\sqrt{-1}\sin\frac{m}{n}\{2(kn+r')\pi+\theta\}$$

$$=\cos\{2km\pi+\frac{m}{n}(2r'\pi+\theta)\}+\sqrt{-1}\sin\{2km\pi+\frac{m}{n}(2r'\pi+\theta)\}$$

$$=\cos\frac{m}{n}(2r'\pi+\theta)+\sqrt{-1}\sin\frac{m}{n}(2r'\pi+\theta),$$

which is one of the values before obtained.

COR. 1. If $m = 1$, then

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{n}}=\cos\frac{2r\pi+\theta}{n}+\sqrt{-1}\sin\frac{2r\pi+\theta}{n}.$$

COR. 2. Since $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}$ may be regarded as expressing, (i) $\{(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{n}}\}^m$, or (ii) $\{(\cos\theta+\sqrt{-1}\sin\theta)^m)^{\frac{1}{n}}$, that is,

$$\text{(i)}\quad\left(\cos\frac{2r\pi+\theta}{n}+\sqrt{-1}\sin\frac{2r\pi+\theta}{n}\right)^m=\cos\frac{m}{n}(2r\pi+\theta)+\sqrt{-1}\sin\frac{m}{n}(2r\pi+\theta),$$

$$\text{or (ii)}\quad(\cos m\theta+\sqrt{-1}\sin m\theta)^{\frac{1}{n}}=\cos\frac{2r\pi+m\theta}{n}+\sqrt{-1}\sin\frac{2r\pi+m\theta}{n},$$

the two values thus resulting must be identical. And this is easily shewn to be the case: for, taking r from 1 to $(n-1)$ in the former, the multiples of 2π will be $\frac{m}{n}, \frac{2m}{n}, \frac{3m}{n}, \ldots \frac{(n-1)m}{n}$; and, if the division by n be performed, no two of these fractions can give the same remainder; since, if $\frac{pm}{n}$ and $\frac{qm}{n}$ were to do this, then their difference $\frac{(p-q)m}{n}$ would be an integer, which is impossible, since p, q, are each less than n, and m prime to it. Hence the $(n-1)$ remainders, being all different and each less than n, will include all the terms of the series 1, 2, 3, ... $(n-1)$; and, consequently, suppressing the multiples of 2π, the n values of the former expression will be identical with those of the latter.

5. *To express* $\sin n\theta$ *and* $\cos n\theta$ *in terms of* θ.

By Demoivre's Theorem, we have

$$\cos n\theta+\sqrt{-1}\sin n\theta=(\cos\theta+\sqrt{-1}\sin\theta)^n=\cos^n\theta\,(1+\sqrt{-1}\tan\theta)^n$$

$$=\cos^n\theta\,(1+n\sqrt{-1}\tan\theta-\frac{n(n-1)}{1.2}\tan^2\theta-\frac{n(n-1)(n-2)}{1.2.3}\sqrt{-1}\tan^3\theta+\&c)$$

hence, equating possible and impossible parts of this expression,

$$\cos n\theta = \cos^n\theta \left\{1 - \frac{n(n-1)}{1.2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tan^4\theta - \&c\right\},$$

$$\sin n\theta = \cos^n\theta \left\{n\tan\theta - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \&c\right\}.$$

Hence also $\tan n\theta = \dfrac{\sin n\theta}{\cos n\theta}$ may be found in terms of $\tan\theta$.

6. **Euler's Series.** *To shew that*

$$\cos a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \&c, \quad \sin a = a - \frac{a^3}{1.2.3} + \&c.$$

In the results of [5], put $n\theta = a$, or $n = \dfrac{a}{\theta}$;

$$\cos a = \cos^n\theta \left\{1 - \frac{a(a-\theta)}{1.2}\left(\frac{\tan\theta}{\theta}\right)^2 + \frac{a(a-\theta)(a-2\theta)(a-3\theta)}{1.2.3.4}\left(\frac{\tan\theta}{\theta}\right)^4 - \&c\right\},$$

$$\sin a = \cos^n\theta \left\{a\left(\frac{\tan\theta}{\theta}\right) - \frac{a(a-\theta)(a-2\theta)}{1.2.3}\left(\frac{\tan\theta}{\theta}\right)^3 + \&c\right\}.$$

Now, while a remains unaltered, suppose n to be increased, and θ to be diminished, indefinitely; then $\left(\dfrac{\tan\theta}{\theta}\right)$ tends to 1 as its limiting value, and

$$\cos^n\theta = (1 - \sin^2\theta)^{\frac{n}{2}} = 1 - \frac{n}{2}\sin^2\theta + \&c = 1 - \frac{a\theta}{2}\left(\frac{\sin\theta}{\theta}\right)^2 + \&c.$$

and, therefore, tends also to 1 as its Limit; hence ultimately, when n is made infinite and θ vanishes, we have

$$\cos a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \&c, \quad \sin a = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \&c.$$

From these results we may obtain, by common division,

$$\tan a = a + \frac{a^3}{3} + \frac{2a^5}{15} + \&c,$$

where, however, the Law of the series is not evident.

7. *To obtain exponential expressions for* $\sin a$, $\cos a$, *and* $\tan a$.

Since (146) $e^x = 1 + x + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} + \dfrac{x^4}{1.2.3.4} + \&c,$

$$\therefore\ e^{a\sqrt{-1}} = 1 + a\sqrt{-1} - \frac{a^2}{1.2} - \frac{a^3\sqrt{-1}}{1.2.3} + \frac{a^4}{1.2.3.4} + \&c,$$

$$e^{-a\sqrt{-1}} = 1 - a\sqrt{-1} - \frac{a^2}{1.2} + \frac{a^3\sqrt{-1}}{1.2.3} + \frac{a^4}{1.2.3.4} - \&c.$$

hence $e^{a\sqrt{-1}} + e^{-a\sqrt{-1}} = 2\left(1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \&c\right) = 2\cos a$,

$$e^{a\sqrt{-1}} - e^{-a\sqrt{-1}} = 2\sqrt{-1}\left(a - \frac{a^3}{1.2.3} + \&c\quad\right) = 2\sqrt{-1}\sin a;$$

and, therefore, $\cos a = \tfrac{1}{2}(e^{a\sqrt{-1}} + e^{-a\sqrt{-1}})$, $\sqrt{-1}\sin a = \tfrac{1}{2}(e^{a\sqrt{-1}} - e^{-a\sqrt{-1}})$,

and $\therefore$ also $\sqrt{-1}\tan a = \dfrac{e^{a\sqrt{-1}} - e^{-a\sqrt{-1}}}{e^{a\sqrt{-1}} + e^{-a\sqrt{-1}}} = \dfrac{e^{2a\sqrt{-1}} - 1}{e^{2a\sqrt{-1}} + 1}$.

COR. Observe that

$$\cos a + \sqrt{-1}\sin a = e^{a\sqrt{-1}},\quad \cos a - \sqrt{-1}\sin a = e^{-a\sqrt{-1}}.$$

8. GREGORY'S SERIES. *To shew that*

$$a = \tan a - \tfrac{1}{3}\tan^3 a + \tfrac{1}{5}\tan^5 a - \&c.$$

From [7] we get

$$e^{2a\sqrt{-1}} = \frac{1 + \sqrt{-1}\tan a}{1 - \sqrt{-1}\tan a},\ \text{or}\ 2a\sqrt{-1} = \log_e \frac{1 + \sqrt{-1}\tan a}{1 - \sqrt{-1}\tan a}:$$

$$= 2\{\sqrt{-1}\tan a + \tfrac{1}{3}(\sqrt{-1}\tan a)^3 + \tfrac{1}{5}(\sqrt{-1}\tan a)^5 + \&c\}\ \text{by (150)}$$

$$= 2\sqrt{-1}\,\{\tan a - \tfrac{1}{3}\tan^3 a + \tfrac{1}{5}\tan^5 a - \&c\};$$

$$\therefore a = \tan a - \tfrac{1}{3}\tan^3 a + \tfrac{1}{5}\tan^5 a - \&c.$$

N.B. Observe that a is here the circular measure of the *least* angle which has the given tangent: for $\dfrac{a}{\tan a} = 1 - \tfrac{1}{3}\tan^2 a + \&c$, where the value of the right side tends to 1 as $\tan a$ tends to *zero;* and this is true also of the left side, if a tends to zero, that is, if a (whether positive or negative) be less than 90°, and so be reduced also to zero, when $\tan a = 0$. The same remark applies to the series for the sine, cosine, and tangent in [6].

COR. If $\tan a = x$, and, therefore, $a = \tan^{-1}x$, the preceding result may be written

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \&c.$$

9. Hence we may arrive at the value of π: for $\tan\tfrac{1}{4}\pi = 1$, and

$$\therefore \frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c = \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \&c.$$

Again, if $\tan a = \tfrac{1}{2}$ and $\tan\beta = \tfrac{1}{3}$, then (86 Ex. 2) $\tan(a + \beta) = 1$;

$$\therefore \tfrac{1}{4}\pi = a + \beta = \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3}$$

$$= \{\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{5}\cdot\frac{1}{2^5} - \&c\} + \{\frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} - \&c\}$$

$$= (\frac{1}{2} + \frac{1}{3}) - \frac{1}{3}(\frac{1}{2^3} + \frac{1}{3^3}) + \frac{1}{5}(\frac{1}{2^5} + \frac{1}{3^5}) - \&c.$$

But the above series do not converge so rapidly as the following, which is very convenient for the calculation of π.

10. MACHIN'S SERIES. *To shew that*

$$\tfrac{1}{4}\pi = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239}.$$

$$\tan^{-1} 1 - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{1-\frac{1}{5}}{1+\frac{1}{5}} = \tan^{-1}\frac{2}{3}$$

$$\tan^{-1}\frac{2}{3} - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{2}{3}-\frac{1}{5}}{1+\frac{2}{15}} = \tan^{-1}\frac{7}{17}$$

$$\tan^{-1}\frac{7}{17} - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{7}{17}-\frac{1}{5}}{1+\frac{7}{85}} = \tan^{-1}\frac{9}{46}$$

$$\tan^{-1}\frac{9}{46} - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{9}{46}-\frac{1}{5}}{1+\frac{9}{230}} = \tan^{-1}\frac{-1}{239};$$

herefore, adding these, we get

an$^{-1}1 - 4\tan^{-1}\frac{1}{5} = \tan^{-1}(-\frac{1}{239}) = -\tan^{-1}\frac{1}{239}$, or $\frac{1}{4}\pi = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$.

11. Hence $\dfrac{\pi}{4} = 4\left\{\dfrac{1}{5} - \dfrac{1}{3}\times\dfrac{1}{5^3} + \dfrac{1}{5}\times\dfrac{1}{5^5} - \dfrac{1}{7}\times\dfrac{1}{5^7} + \&c\right\} - \left\{\dfrac{1}{239} - \&c\right\}$,

rom which the value of π may be calculated as follows:

$$\frac{1}{5} = .2 \qquad\qquad \frac{1}{3}\times\frac{1}{5^3} = \frac{2^3}{3\times10^3} = \frac{.008}{3} = .0026666$$

$$\frac{1}{5^5} = \frac{1}{5^6} = \frac{2^6}{10^6} = \frac{64}{10^6} = \underline{.000064} \qquad \frac{1}{7}\times\frac{1}{5^7} = \frac{2^7}{7\times10^7} = \frac{.0000128}{7} = \underline{.0000018}$$

$$\begin{array}{rr}
 & .200064 \qquad\qquad\qquad\qquad .0026684 \\
\text{subtract} & \underline{.0026684} \\
 & .1973956 \\
 & \underline{4} \\
 & .7895824 \\
\text{subtract } \frac{1}{239} = & \underline{.0041841} \\
 & .78539\,83 = \frac{1}{4}\pi \\
 & \underline{4} \\
 & 3.14159|32 = \pi,\ \text{accurately to five places.}
\end{array}$$

CHAPTER II.

ON CERTAIN OTHER TRIGONOMETRICAL SERIES.

12. *To investigate Euler's series for* $\sin\theta$ *and* $\cos\theta$ *i terms of* θ, *without the aid of Demoivre's Theorem.*

Since $\sin(-\theta) = -\sin\theta$, whereas $\cos(-\theta) = \cos\theta$, we see at on that the series for $\sin\theta$ in terms of θ can only contain *odd* powe of θ, and the series for $\cos\theta$ only *even* powers of θ; since th former will be changed in sign but not in value, and the latter wi not be changed at all, by writing in it $-\theta$ for θ.

Also when $\theta = 0$, $\sin\theta = 0$, and $\cos\theta = 1$: hence every term the series for $\sin\theta$ must contain θ, but in that for $\cos\theta$ the must be a term 1, not containing θ.

Assume then

$$\sin\theta = a_1\theta + a_3\theta^3 + a_5\theta^5 + \&c, \quad \cos\theta = 1 + a_2\theta^2 + a_4\theta^4 + \&c.$$

Hence $\dfrac{\sin\theta}{\theta} = a_1 + a_3\theta^2 + a_5\theta^4 + \&c$, or, putting $\theta = 0$, $\dfrac{\sin\theta}{\theta} = 1 = a$

putting this value for a_1, and adding the two series, we have

$$1 + \theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 + \&c = \sin\theta + \cos\theta.$$

In this result write $\theta + \phi$ for θ; then

$$1 + (\theta+\phi) + a_2(\theta+\phi)^2 + a_3(\theta+\phi)^3 + a_4(\theta+\phi)^4 + \&c.$$
$$= \sin(\theta+\phi) + \cos(\theta+\phi) = \cos\phi(\cos\theta + \sin\theta) + \sin\phi(\cos\theta - \sin\theta$$
$$= (1 + a_2\phi^2 + a_4\phi^4 + \&c)\,(1 + \theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + \&c)$$
$$+ (\phi + a_3\phi^3 + a_5\phi^5 + \&c)\,(1 - \theta + a_2\theta^2 - a_3\theta^3 + a_4\theta^4 - \&c):$$

whence, equating coefficients of $\phi\theta$, $\phi\theta^2$, $\phi\theta^3$, &c, we get

$$2a_2 = -1, \quad 3a_3 = a_2, \quad 4a_4 = -a_3, \quad \&c,$$

$$\text{or} \quad a_2 = -\frac{1}{1.2}, \quad a_3 = -\frac{1}{1.2.3}, \quad a_4 = \frac{1}{1.2.3.4}, \quad \&c.$$

and, therefore,

$$\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \&c, \quad \cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \&c.$$

13. *If* $2\cos a = x + \dfrac{1}{x}$, *then* $2\sqrt{-1}\,\sin a = x - \dfrac{1}{x}$

and also $2\cos ma = x^m + \dfrac{1}{x^m}$, $2\sqrt{-1}\,\sin ma = x^m - \dfrac{1}{x^m}$.

For, if $2\cos a = x + \frac{1}{x}$, then $4\cos^2 a = x^2 + 2 + \frac{1}{x^2} = 4 - 4\sin^2 a$;

$$\therefore\ -4\sin^2 a = x^2 - 2 + \frac{1}{x^2}, \text{ and } 2\sqrt{-1}\sin a = x - \frac{1}{x};$$

hence also $\cos a + \sqrt{-1}\sin a = x$, $\cos a - \sqrt{-1}\sin a = \frac{1}{x}$,

and [3] $\cos ma + \sqrt{-1}\sin ma = x^m$, $\cos ma - \sqrt{-1}\sin ma = \frac{1}{x^m}$;

therefore, $2\cos ma = x^m + \frac{1}{x^m}$, $2\sqrt{-1}\sin ma = x^m - \frac{1}{x^m}$.

It is obvious from [7 Cor.] that the quantity x here introduced is no other than $e^{a\sqrt{-1}}$, but it is often more convenient to use x than $e^{a\sqrt{-1}}$.

Ex. If $2\cos a = x + \frac{1}{x} = x + x^{-1}$, we have

$$2\cos\tfrac{1}{2}a = x^{\frac{1}{2}} + x^{-\frac{1}{2}},\ \ 2\sqrt{-1}\sin 3a = x^3 - x^{-3},\ \&c.$$

14. *If* $2\cos a = x + x^{-1}$, *and* $2\cos\beta = y + y^{-1}$, *then*

$$2\cos(ma + n\beta) = x^m y^n + x^{-m} y^{-n}.$$

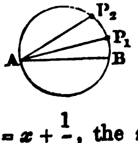
For $2\cos(ma + n\beta)$

$$\begin{aligned}
&= 2(\cos ma \cos n\beta - \sin ma \sin n\beta)\\
&= \tfrac{1}{2}(2\cos ma . 2\cos n\beta + 2\sqrt{-1}\sin ma . 2\sqrt{-1}\sin n\beta)\\
&= \tfrac{1}{2}\{(x^m + x^{-m})(y^n + y^{-n}) + (x^m - x^{-m})(y^n - y^{-n})\} \text{ by [13]}\\
&= x^m y^n + x^{-m} y^{-n}.
\end{aligned}$$

This includes the proof that $2\cos(ma - n\beta) = x^m y^{-n} + x^{-m} y^n$, and also, by [13], that $2\sqrt{-1}\sin(ma + n\beta) = x^m y^n - x^{-m} y^{-n}$, &c: and the same reasoning may be extended to any number of angles.

Ex. If $2\cos\gamma = z + z^{-1}$, then $2\cos(a - 2\beta + \frac{1}{3}\gamma) = xy^{-2}z^{\frac{1}{3}} + x^{-1}y^2z^{-\frac{1}{3}}$.

15. VIETA's *Property of Chords.*



Let AB be the diameter of a circle, and BP_1, P_1P_2, P_2P_3, &c. circular arcs, each equal to θ; and join AP_1, AP_2, &c: then, if (to radius *unity*) the first of these lines AP_1 be taken $= x + \frac{1}{x}$, the n^{th} line AP_n will $= x^n + \frac{1}{x^n}$.

For $AP_1 = \text{chd}(\pi - \theta) = 2\sin\frac{1}{2}(\pi - \theta) = 2\cos\frac{1}{2}\theta = x + \frac{1}{x}$, by hypothesis;

and $AP_n = \text{chd}(\pi - n\theta) = 2\sin\frac{1}{2}(\pi - n\theta) = 2\cos\frac{1}{2}n\theta = x^n + \frac{1}{x^n}$.

16. *To express* $\cos^n\theta$ *in terms of the cosines of multiples of* θ, *when* n *is a positive integer.*

Put $2\cos\theta = x + x^{-1}$: then we have $(2\cos\theta)^n = (x+x^{-1})^n$

$$= x^n + nx^{n-2} + \frac{n(n-1)}{1.2}x^{n-4} + \&c + \frac{n(n-1)}{1.2}x^{-(n-4)} + nx^{-(n-2)} + x^{-n}\ldots(\alpha)$$

$$= (x^n + x^{-n}) + n(x^{n-2} + x^{-(n-2)}) + \frac{n(n-1)}{1.2}(x^{n-4} + x^{-(n-4)}) + \&c.$$

$$= 2\cos n\theta + n\{2\cos(n-2)\theta\} + \frac{n(n-1)}{1.2}\{2\cos(n-4)\theta\} + \&c:$$

whence, dividing each side by 2, we get

$$2^{n-1}\cos^n\theta = \cos n\theta + \cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta + \&c\ldots(\beta)$$

If n be even, there will be an odd middle term in (α), namely, the $(\frac{1}{2}n+1)$th, whose value will be

$$\frac{n(n-1)\ldots(n-\frac{1}{2}n+1)}{1.2\ldots\frac{1}{2}n}x^{\frac{n}{2}}x^{-\frac{n}{2}} = \frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2\ldots\frac{1}{2}n};$$

and so the last term of (β) will be $\frac{1}{2}\cdot\frac{n(n-1)\ldots(\frac{1}{2}n+1)}{1.2\ldots\frac{1}{2}n}$.

If n be odd, there will be two middle terms in (α), namely, the $\{\frac{1}{2}(n-1)+1\}$th and $\{\frac{1}{2}(n+1)+1\}$th, whose coefficients, however, will be equal, one of these containing the factor $x^{\frac{1}{2}(n+1)}x^{-\frac{1}{2}(n-1)}$ or x, and the other, the factor $x^{\frac{1}{2}(n-1)}x^{-\frac{1}{2}(n+1)}$ or x^{-1}: hence their sum will be

$$\frac{n(n-1)\ldots\{n-\frac{1}{2}(n-1)+1\}}{1.2\ldots\frac{1}{2}(n-1)}(x+x^{-1}) = \frac{n(n-1)\ldots\{\frac{1}{2}(n+3)\}}{1.2\ldots\frac{1}{2}(n-1)}(2\cos\theta);$$

and the last term of (β) will now be $\frac{n(n-1)\ldots\{\frac{1}{2}(n+3)\}}{1.2\ldots\frac{1}{2}(n-1)}\cos\theta$.

Ex. 1. $2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3.$

Ex. 2. $2^4\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta.$

17. *To express* $\sin^n\theta$ *in terms of sines or cosines of multiples of* θ, *when* n *is a positive integer.*

Put $2\cos\theta = x + x^{-1}$;
then $2\sqrt{-1}\sin\theta = x - x^{-1}$, and $(-1)^{\frac{1}{2}n}(2\sin\theta)^n = (x-x^{-1})^n$

$$= x^n - nx^{n-2} + \frac{n(n-1)}{1.2}x^{n-4} - \&c + \frac{n(n-1)}{1.2}(-x)^{-(n-4)} + n(-x)^{-(n-2)} + (-x)^{-n}.$$

(i) If n be even, then the above will become

$$(-1)^{\frac{1}{2}n}(2\sin\theta)^n = (x^n + x^{-n}) - n(x^{n-2} + x^{-(n-2)}) + \&c.$$
$$= 2\cos n\theta - n\{2\cos(n-2)\theta\} + \&c;$$

$$-1)^{\frac{1}{2}n}.2^{n-1}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta - \&c,$$

ere the last term, as in [16], will be

$$.\frac{n(n-1)\dots(\frac{1}{2}n+1)}{1.2\dots\frac{1}{2}n}x^{\frac{n}{2}}(-x)^{-\frac{n}{2}} = (-1)^{\frac{n}{2}}\frac{1}{2}.\frac{n(n-1)\dots(\frac{1}{2}n+1)}{1.2\dots\frac{1}{2}n}.$$

(ii) If n be odd, then the above will become

$$(-1)^{\frac{1}{2}n}(2\sin\theta)^n = (x^n - x^{-n}) - n(x^{n-2} - x^{-(n-2)}) + \&c.$$
$$= 2\sqrt{-1}\sin n\theta - n\{2\sqrt{-1}\sin(n-2)\theta\} + \&c;$$

$$(-1)^{\frac{1}{2}(n-1)}2^{n-1}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1.2}\sin(n-4)\theta - \&c,$$

ere the last term will be

$$\frac{1}{2}.\frac{n(n-1)\dots\frac{1}{2}(n+3)}{1.2\dots\frac{1}{2}(n-1)}\{x^{\frac{n+1}{2}}(-x)^{-\frac{n-1}{2}} + x^{\frac{n-1}{2}}(-x)^{-\frac{n+1}{2}}\}$$
$$= (-1)^{\frac{n-1}{2}}.\frac{1}{2}.\frac{n(n-1)\dots\frac{1}{2}(n+3)}{1.2\dots\frac{1}{2}(n-1)}(x - x^{-1})$$
$$= (-1)^{\frac{n-1}{2}}.\frac{n(n-1)\dots\frac{1}{2}(n+3)}{1.2\dots\frac{1}{2}(n-1)}\sin\theta.$$

Ex. 1. $-2^5\sin^6\theta = \sin 6\theta - 6\sin 4\theta + 15\sin 2\theta - 10.$

Ex. 2. $+2^4\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta.$

N.B. For the complete forms of the above series for $\cos^n\theta$ and $^n\theta$, for *all* values of n, reference may be made to Peacock's *ebra*, Vol. II, Artt. 1055-1060.

18. *To express* cos$n\theta$ *in descending powers of* cosθ, *en* n *is a positive integer.*

Put $2\cos\theta = x + \frac{1}{x} = y$, suppose; $\therefore 2\cos n\theta = x^n + \frac{1}{x^n}$:

w $(1 - xz)(1 - \frac{z}{x}) = 1 - (x + \frac{1}{x})z + z^2 = 1 - yz + z^2 = 1 - z(y - z)$;

ence, taking logarithms on both sides, we get

$$\log(1 - xz) + \log(1 - \frac{z}{x}) = \log\{1 - z(y - z)\},\text{ or}$$

$$+\frac{x^2z^2}{2} + \&c + (\frac{z}{x} + \frac{z^2}{2x^2} + \&c) = z(y-z) + \frac{z^2(y-z)^2}{n} + \&c + \frac{z^n(y-z)^n}{2} + \&c.$$

Now the coefficient of z^n in $\dfrac{z^{n-r}(y-z)^{n-r}}{n-r}$ is

$$\frac{(-1)^r}{n-r}\cdot\frac{(n-r)(n-r-1)\dots(n-2r+1)}{1.2\dots r}\,y^{n-2r};$$

$\therefore$ equating coefficients of z^n on both sides of the above equation,

$$\frac{1}{n}\left(x^n+\frac{1}{x^n}\right)=\Sigma\left\{\frac{(-1)^r}{n-r}\cdot\frac{(n-r)(n-r-1)\dots(n-2r+1)}{1.2\dots r}\,y^{n-2r}\right\},\text{ or}$$

$$2\cos n\theta=n\,\Sigma\left\{\frac{(-1)^r}{n-r}\cdot\frac{(n-r)(n-r-1)\dots(n-2r+1)}{1.2\dots r}\,(2\cos\theta)^{n-2r}\right\},$$

where r is to be taken from 0 to $\frac{1}{2}n$ if n is even, or to $\frac{1}{2}(n-1)$ if n is odd.

Hence $2\cos n\theta$

$$=(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\frac{n(n-3)}{1.2}(2\cos\theta)^{n-4}-\&c.$$

19. *To express* $\sin n\theta$ *and* $\cos n\theta$ *in ascending powers of* $\sin\theta$ *and* $\cos\theta$, *for all values of* n.

Assume $\sin n\theta=A_0+A_1\sin\theta+A_2\sin^2\theta+\&c+A_p\sin^p\theta+\&c\dots;(\alpha)$ then, writing $\theta+h$ for θ, we have

$$\sin n(\theta+h)=A_0+A_1\sin(\theta+h)+A_2\sin^2(\theta+h)+\&c.$$
$$=A_0+A_1\{\sin\theta(1-\tfrac{1}{2}h^2)+h\cos\theta\}+\&c,$$

(expanding $\cos h$ and $\sin h$ by [6], and retaining only throughout the investigation the terms involving h and h^2,)

$$=A_0+A_1(\sin\theta+h\cos\theta-\tfrac{1}{2}h^2\sin\theta)+A_2(\sin\theta+h\cos\theta-\tfrac{1}{2}h^2\sin\theta)^2+\&c.$$
$$+A_p(\sin\theta+h\cos\theta-\tfrac{1}{2}h^2\sin\theta)^p+\&c.$$
$$=A_0+A_1(\sin\theta+h\cos\theta-\tfrac{1}{2}h^2\sin\theta)$$
$$+A_2(\sin^2\theta+2h\sin\theta\cos\theta-h^2\sin^2\theta+h^2\cos^2\theta)+\&c$$
$$+A_p\{\sin^p\theta+ph\sin^{p-1}\theta\cos\theta-\tfrac{1}{2}ph^2\sin^p\theta+\tfrac{1}{2}p(p-1)h^2\sin^{p-2}\theta\cos^2\theta\}+\&c:$$

but $\sin n(\theta+h)=\sin(n\theta+nh)=\sin n\theta\cos nh+\cos n\theta\sin nh$

$$=\sin n\theta(1-\tfrac{1}{2}n^2h^2)+nh\cos n\theta+\&c:$$

therefore, equating powers of h, and $\frac{1}{2}h^2$, we get

$$n\cos n\theta=(A_1+2A_2\sin\theta+\&c+pA_p\sin^{p-1}\theta+\&c)\cos\theta\dots(\beta)$$
$$n^2\sin n\theta=A_1\sin\theta+2A_2\sin^2\theta+\&c+pA_p\sin^p\theta+\&c$$
$$-\cos^2\theta\{2.1A_2+\&c+p(p-1)A_p\sin^{p-2}\theta+(p+1)pA_{p+1}\sin^{p-1}\theta$$
$$+(p+2)(p+1)A_{p+2}\sin^p\theta+\&c\}\dots\dots(\gamma).$$

In the last result, write $1-\sin^2\theta$ for $\cos^2\theta$, and compare the coefficient of $\sin^p\theta$, with its value in $n^2\times(a)$: thus we get

$$pA_p+p(p-1)A_p-(p+2)(p+1)A_{p+2}=n^2A_p,$$

$$\text{or}\quad A_{p+2}=-\frac{n^2-p^2}{(p+1)(p+2)}A_p;$$

from which, giving p the successive values 0, 2, 4, &c, we get

$$A_2=-\frac{n^2}{1.2}A_0,\qquad A_4=-\frac{(n^2-2^2)}{3.4}A_2=+\frac{n^2(n^2-2^2)}{1.2.3.4}A_0,\ \&c;$$

and so, giving p the successive values 1, 3, 5, &c, we get

$$A_3=-\frac{n^2-1^2}{2.3}A_1,\quad A_5=-\frac{n^2-3^2}{4.5}A_3=+\frac{(n^2-1^2)(n^2-3^2)}{2.3.4.5}A_1,\ \&c.$$

To find A_0 and A_1, put $\theta=r\pi$ in (a) and (β), r being any integer; then

$$A_0=\sin nr\pi,\ A_1=n\frac{\cos nr\pi}{\cos r\pi}=n\frac{\cos\{(n-1)r\pi+r\pi\}}{\cos r\pi}=n\cos(n-1)r\pi:$$

hence (a) $\sin n\theta=(A_0+A_2\sin^2\theta+\&c)+(A_1\sin\theta+A_3\sin^3\theta+\&c)$

$$=\sin nr\pi\left(1-\frac{n^2}{1.2}\sin^2\theta+\frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta-\&c\right)$$

$$+\cos(n-1)r\pi\left(\frac{n}{1}\sin\theta-\frac{n(n^2-1^2)}{1.2.3}\sin^3\theta+\frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\sin^5\theta-\&c\right):(\delta)$$

and (β) $n\cos n\theta=(A_1+3A_3\sin^2\theta+\&c)\cos\theta$

$$+(2A_2\sin\theta+4A_4\sin^3\theta+\&c)\cos\theta,$$

$$\text{or}\ \cos n\theta=\cos(n-1)r\pi\left(1-\frac{n^2-1^2}{1.2}\sin^2\theta+\frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\sin^4\theta-\&c\right)\cos\theta$$

$$-\sin nr\pi\left(n\sin\theta-\frac{n(n^2-2^2)}{1.2}\sin^3\theta+\frac{n(n^2-2^2)(n^2-4^2)}{1.2.3.4}\sin^5\theta-\&c\right)\cos\theta.\ ($$

The above results are true for all values of n: but if n be an *odd integer*, then $\sin nr\pi=0$, and $\cos(n-1)r\pi=1$;

$$\therefore\ \sin n\theta=n\sin\theta-\frac{n(n^2-1^2)}{1.2.3}\sin^3\theta+\frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\sin^5\theta\text{–}\&c,\ n\ \textit{odd},\ (1$$

$$\text{and}\ \cos n\theta=\left(1-\frac{n^2-1^2}{1.2}\sin^2\theta+\frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\sin^4\theta-\&c\right)\cos\theta,\ \ldots\ldots\ldots(2$$

where the series in each case will terminate, as soon as the factor n^2-n^2 enters the numerator.

If, however, n be an *even integer*, then, as before, one line in the values of $\sin n\theta$ and $\cos n\theta$ will disappear, but the series in the other line will never terminate, the quantities 1^2, 3^2, &c. in the numerator being all *odd*, and n^2 even.

Exactly in the same manner, by assuming

$$\cos n\theta = A_0 + A_1 \sin\theta + A_2 \sin^2\theta + \&c,$$

we may shew that

$$\cos n\theta = \cos nr\pi \left(1 - \frac{n^2}{1.2}\sin^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta - \&c\right)$$

$$- \sin(n-1)r\pi \left(n\sin\theta - \frac{n(n^2-1^2)}{1.2.3}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\sin^5\theta - \&c\right);$$

and $\sin n\theta = \cos nr\pi \left(n\sin\theta - \dfrac{n(n^2-2^2)}{1.2.3.4}\sin^3\theta + \&c\right)\cos\theta$

$$+ \sin(n-1)\,r\pi\left(1 - \frac{n^2-1^2}{1.2}\sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\sin^4\theta - \&c\right)\cos\theta.$$

If n be an integer the second line in each case disappears; and, if n be also even, the first line terminates, and we have

$$\cos n\theta = 1 - \frac{n^2}{1.2}\sin^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta - \&c, \; n \textit{ even}, \ldots\ldots (3)$$

$$\sin n\theta = \left(n\sin\theta - \frac{n(n^2-2^2)}{1.2.3.4}\sin^3\theta + \&c\right)\cos\theta, \; \ldots\ldots\ldots\ldots\ldots (4)$$

Lastly, writing $\frac{1}{2}\pi - \theta$ for θ in the four above results, we get

$$(-1)^{\frac{1}{2}(n-1)}\cos n\theta = n\cos\theta - \frac{n(n^2-1^2)}{1.2.3}\cos^3\theta + \&c \; (n \textit{ odd}) \ldots\ldots\ldots\ldots (5)$$

$$(-1)^{\frac{1}{2}(n-1)}\sin n\theta = \left(1 - \frac{n^2-1^2}{1.2}\cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\cos^4\theta - \&c\right)\sin\theta \; \ldots\ldots (6)$$

$$(-1)^{\frac{1}{2}n}\cos n\theta = 1 - \frac{n^2}{1.2}\cos^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\cos^4\theta - \&c \; (n \textit{ even}) \ldots (7)$$

$$(-1)^{\frac{1}{2}n+1}\sin n\theta = \left(n\cos\theta - \frac{n(n^2-2^2)}{1.2.3.4}\cos^3\theta + \&c\right)\sin\theta \ldots\ldots\ldots\ldots\ldots (8)$$

N.B. The Student, who is acquainted with the Differential Calculus, will see that equations (β) and (γ) may be obtained at once by merely twice differentiating (α) with respect to θ; and, (δ) being thus obtained, (ϵ) also follows from it by differentiation: and, similarly, for the latter set of equations.

CHAPTER III.

ON THE SOLUTION OF EQUATIONS BY TRIGONOMETRY.

20. *To solve a quadratic equation by Trigonometry.*

(i) Let $x^2+px-q=0$; then $x=-\frac{1}{2}p\pm\frac{1}{2}\sqrt{(p^2+4q)}$:

Put $4q=p^2\cot^2\theta$, or $\frac{p}{2\sqrt{q}}=\cot\theta$, which is always possible, since $\cot\theta$ may have any sign or magnitude: then

$$x=-\sqrt{q}\,(\cot\theta\pm\operatorname{cosec}\theta)=-\sqrt{q}\cot\tfrac{1}{2}\theta \text{ or } -\sqrt{q}\tan\tfrac{1}{2}\theta.$$

Hence $\quad L\cot\theta=10+\log p-\log 2-\frac{1}{2}\log q$;

$$\log(-x)=\tfrac{1}{2}\log q+L\cot\tfrac{1}{2}\theta-10, \text{ or } =\tfrac{1}{2}\log q+L\tan\tfrac{1}{2}\theta-10$$

The same reasoning will serve for the equation $x^2-px-q=0$.

(ii) Let $x^2+px+q=0$; then $x=-\frac{1}{2}p\pm\frac{1}{2}\sqrt{(p^2-4q)}$: and here, if the roots are possible, or $p^2>4q$, put $4q=p^2\sin^2\theta$, which is always possible, since $p>2\sqrt{q}$, and, therefore, $\sin\theta<1$; then

$$x=-\tfrac{1}{2}p\pm\tfrac{1}{2}\sqrt{(p^2-p^2\sin^2\theta)}=-\tfrac{1}{2}p\{1\pm\cos\theta\}=-p\cos^2\tfrac{1}{2}\theta, \text{ or } -p\sin^2\tfrac{1}{2}\theta.$$

Hence $\quad L\operatorname{cosec}\theta=10+\log p-\log 2-\frac{1}{2}\log q$;

$$\log(-x)=\log p+2L\cos\tfrac{1}{2}\theta-20, \text{ or } =\log p+2L\sin\tfrac{1}{2}\theta-20.$$

But, if $p^2<4q$, put $4q=p^2\sec^2\theta$; then

$$x=-\tfrac{1}{2}p\pm\tfrac{1}{2}\sqrt{(p^2-p^2\sec^2\theta)}=\tfrac{1}{2}p\{-1\pm\sqrt{-1}\tan\theta\}.$$

The same reasoning will apply to the equation $x^2-px+q=0$.

N.B. The above results are useful, when the quantities p and q happen to be numbers of many figures.

21. *To solve a cubic equation by Trigonometry.*

Let $x^3+ax^2+bx+c=0$ be a cubic equation: by writing $x'-\frac{1}{3}a$ for x, this may be always reduced to the form

$$x'^3-(\tfrac{1}{3}a^2-b)\,x'-(\tfrac{1}{3}ab-\tfrac{2}{27}a^3-c)=0,$$

which wants its second term.

Hence we may assume $x^3-qx-r=0$ as the type of a cubic equation, where q and r may be positive or negative.

For x write $\frac{y}{n}$; then $y^3-qn^2y-rn^3=0$: but we know that

$$\cos 3\phi=4\cos^3\phi-3\cos\phi, \text{ or } \cos^3\phi-\tfrac{3}{4}\cos\phi-\tfrac{1}{4}\cos 3\phi=0;$$

and, if we put $qn^2=\frac{3}{4}$, $rn^3=\frac{1}{4}\cos 3\phi$, these two equations will become identical, that is, we shall have $y=\cos\phi$. Hence we get

$$n^2=\frac{3}{4q}, \text{ and } \cos 3\phi=4rn^3=4r\left(\frac{3}{4q}\right)^{\frac{3}{2}}=\frac{r}{2}\sqrt{\frac{27}{q^3}}.$$

Let a be the *least* value of 3ϕ, determined as above: then, since $\cos 3\phi = \cos(2\pi \pm a)$, we get the three values of $\cos\phi$ to be $\cos\frac{1}{3}a$ and $\cos\frac{1}{3}(2\pi \pm a)$; and, therefore, those of $\frac{y}{n}$, or x, will be $2\sqrt{\frac{q}{3}}\cos\frac{a}{3}$, and $2\sqrt{\frac{q}{3}}\cos\frac{2\pi \pm a}{3}$, where $\cos a = \frac{r}{2}\sqrt{\frac{27}{q^3}}$.

Since $\cos a < 1$, we must have $\frac{r}{2}\sqrt{\frac{27}{q^3}} < 1$, or $\frac{r^2}{4} < \frac{q^3}{27}$; that is, the above method can only be applied when *Cardan's* method of solution, which is given below, is found to fail.

22. *To adapt the roots of Cardan's Solution for Logarithmic computation.*

In the equation $x^3 - qx - r = 0$, put $x = y + z$; then we get

$$y^3 + z^3 + (3yz - q)(y + z) - r = 0:$$

assume $3yz - q = 0$, and, therefore, $y^3 + z^3 = r$, where $y^3z^3 = \frac{1}{27}q^3$: hence y^3 and z^3 are the roots of the quadratic $v^2 - rv + \frac{1}{27}q^3 = 0$,

so that $$y^3 = \frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)},\quad z^3 = \frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)},$$

and $$x = y + z = \sqrt[3]{\left\{\frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}} + \sqrt[3]{\left\{\frac{r}{2} - \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)}\right\}}.$$

The above is *Cardan's Solution*, where, however, if $\frac{r^2}{4} < \frac{q^3}{27}$, the roots, though really possible, appear under an impossible form, and the method is said to fail.

(i) To this case the reasoning of the last Article applies; and the roots, there obtained, may be easily derived from the above. For put $\frac{r^2}{4} = \frac{q^3}{27}\cos^2 a$, which is always possible, since $\frac{r^2}{4} < \frac{q^3}{27}$;

then $$x = \sqrt{\frac{q}{3}}\left\{(\cos a + \sqrt{-1}\sin a)^{\frac{1}{3}} + (\cos a - \sqrt{-1}\sin a)^{\frac{1}{3}}\right\}$$

$$= 2\sqrt{\frac{q}{3}}\cos\frac{a}{3} \text{ or } 2\sqrt{\frac{q}{3}}\cos\frac{2\pi \pm a}{3},$$

as appears easily by Demoivre's Theorem.

(ii) If, however, $\frac{r^2}{4} > \frac{q^3}{27}$, then, if the original equation be of the form $x^3 - qx - r = 0$, where the coefficient of x is really negative, put $\frac{r^2}{4} = \frac{q^3}{27}\operatorname{cosec}^2\theta$: then we have

$$x = \sqrt{\frac{q}{3}}\,\{\sqrt[3]{(\operatorname{cosec}\theta + \cot\theta)} + \sqrt[3]{(\operatorname{cosec}\theta - \cot\theta)}\}$$

$$= \sqrt{\frac{q}{3}}\,\{\sqrt[3]{\cot\tfrac{1}{2}\theta} + \sqrt[3]{\tan\tfrac{1}{2}\theta}\} = \sqrt{\frac{q}{3}}\,(\cot\phi + \tan\phi),\ \text{if } \tan\phi = \sqrt[3]{\tan\tfrac{1}{2}\theta},$$

$$= \sqrt{\frac{q}{3}} \cdot \frac{1}{\sin\phi\cos\phi} = 2\sqrt{\frac{q}{3}}\operatorname{cosec}2\phi.$$

But, if the original equation be of the form $x^3 + qx - r = 0$, where the coefficient of x is positive, we should get in the same way

$$x = \sqrt[3]{\{\frac{r}{2} + \sqrt{(\frac{r^2}{4} + \frac{q^3}{27})}\}} + \sqrt[3]{\{\frac{r}{2} - \sqrt{(\frac{r^2}{4} + \frac{q^3}{27})}\}};$$

and now put $\frac{r^2}{4} = \frac{q^3}{27}\cot^2\theta$: then, as before, we obtain

$$x = \sqrt{\frac{q}{3}}\,\{\sqrt[3]{(\cot\theta + \operatorname{cosec}\theta)} + \sqrt[3]{(\cot\theta - \operatorname{cosec}\theta)}\}$$

$$= \sqrt{\frac{q}{3}}\,(\sqrt[3]{\cot\tfrac{1}{2}\theta} - \sqrt[3]{\tan\tfrac{1}{2}\theta}) = \sqrt{\frac{q}{3}}\,(\cot\phi - \tan\phi) = 2\sqrt{\frac{q}{3}}\cot 2\phi.$$

23. *To find the roots of $x^n - 1 = 0$, and to resolve $x^n - 1$ into factors.*

If $x^n - 1 = 0$, then we have $x^n = 1 = \cos 2r\pi \pm \sqrt{-1}\sin 2r\pi$;

$$\therefore\ x = \cos\frac{2r\pi}{n} \pm \sqrt{-1}\sin\frac{2r\pi}{n}.$$

From this expression, by giving r all values from 0 to $\frac{1}{2}n$ if n be even, or to $\frac{1}{2}(n-1)$ if n be odd, we shall obtain n values for x, the *first* pair of roots being each $= +1$, in both cases, and the *last* pair being each $= -1$, when n is *even*.

Now, excepting those just mentioned, all these values will be different from one another: for, suppose that

$$\cos\frac{2p\pi}{n} + \sqrt{-1}\sin\frac{2p\pi}{n} = \cos\frac{2q\pi}{n} \pm \sqrt{-1}\sin\frac{2qp}{n}:$$

then we get $\cos\frac{2p\pi}{n} = \cos\frac{2q\pi}{n}$, and $\sin\frac{2p\pi}{n} = \pm\sin\frac{2q\pi}{n}$;

$$\text{hence}\ \frac{2p\pi}{n} = 2m\pi \pm \frac{2q\pi}{n},\ \text{or}\ \frac{p \mp q}{n} = m,\ \text{an integer.}$$

Hence $p \mp q$ is a multiple of n, which is impossible, since p and q are each less than $\frac{1}{2}n$: and, consequently, the n values of x, obtained as above, will be all different from one another.

And, further, we can get no new values for x by taking for r a value r', which is greater than $\frac{1}{2}n$. For any integer r' may be put into the *form* $mn \pm r$, where r is not greater than $\frac{1}{2}n$; and

$$\cos\frac{2r'\pi}{n}+\sqrt{-1}\sin\frac{2r'\pi}{n}=\cos(2m\pi\pm\frac{2r\pi}{n})+\sqrt{-1}\sin(2m\pi\pm\frac{2r\pi}{n})$$
$$=\cos\frac{2r\pi}{n}\pm\sqrt{-1}\sin\frac{2r\pi}{n},$$

each of which values is included among those already obtained, since r is not greater than $\frac{1}{2}n$: and similarly for

$$\cos\frac{2r'\pi}{n}-\sqrt{-1}\sin\frac{2r'\pi}{n}.$$

We obtain, therefore, the n roots of the equation $x^n-1=0$ from the above expression, by putting $r=0, 1, 2,$ &c. successively.

Now (i) if n be even, these values of x are

$$+1,\ \cos\frac{2\pi}{n}\pm\sqrt{-1}\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n}\pm\sqrt{-1}\sin\frac{4\pi}{n},\ \&c,$$
$$\cos\frac{(n-2)\pi}{n}\pm\sqrt{-1}\sin\frac{(n-2)\pi}{n},\ -1,$$

the first and last values arising when $r=0$ and when $r=\frac{1}{2}n$; hence, by the Theory of Equations, (observing that

$$\{x-(\cos\theta+\sqrt{-1}\sin\theta)\}\{x-(\cos\theta-\sqrt{-1}\sin\theta)\}$$
$$=\{(x-\cos\theta)-\sqrt{-1}\sin\theta\}\{(x-\cos\theta)+\sqrt{-1}\sin\theta\}$$
$$=(x-\cos\theta)^2+\sin^2\theta=x^2-2x\cos\theta+1,)$$

we have $x^n-1=(x^2-1)(x^2-2x\cos\frac{2\pi}{n}+1)(x^2-2x\cos\frac{4\pi}{n}+1)\ldots$

So (ii), if n be odd, we shall get

$$(x^n-1)=(x-1)(x^2-2x\cos\frac{2\pi}{n}+1)(x^2-2x\cos\frac{4\pi}{n}+1)\ldots$$

COR. Instead of using the double sign, we might solve the above equation as follows:

$$x^n=1=\cos 2r\pi+\sqrt{-1}\sin 2r\pi,\text{ and } x=\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n},$$

where r may be taken from 1 to n.

These values of x will, of course, be identical with the former: thus, if $r<n$,

$$\cos\frac{2(n-r)\pi}{n}+\sqrt{-1}\sin\frac{2(n-r)\pi}{n}=\cos(2\pi-\frac{r\pi}{n})+\sqrt{-1}\sin(2\pi-\frac{r\pi}{n})$$
$$=\cos\frac{r\pi}{n}-\sqrt{-1}\sin\frac{r\pi}{n},$$

one of the roots which would be found in [23].

Hence the different values of x may be expressed by the series

$$\cos\frac{2\pi}{n}+\sqrt{-1}\,\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n}+\sqrt{-1}\,\sin\frac{4\pi}{n},\ \cos\frac{6\pi}{n}+\sqrt{-1}\,\sin\frac{6\pi}{n},\ \&c;$$

or, denoting the *first* of them by a, it follows, by Demoivre's Theorem, that they may all be expressed by the series $a, a^2, a^3, \ldots a^n$ or 1.

Ex. If $x^3-1=0$, the root (a) of x will be

$$\cos\tfrac{2}{3}\pi+\sqrt{-1}\,\sin\tfrac{2}{3}\pi=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\,\sqrt{-1};$$

and the other two roots will be

$$(-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\,\sqrt{-1})^2=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3}\,\sqrt{-1},\text{ and }(-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\,\sqrt{-1})^3=1.$$

24. *To find the roots of $x^n+1=0$, and to resolve x^n+1 into factors.*

If $x^n+1=0$, then $x^n=-1=\cos(2r+1)\pi\pm\sqrt{-1}\,\sin(2r+1)\pi$;

$$\text{and, consequently, } x=\cos\frac{(2r+1)\pi}{n}\pm\sqrt{-1}\,\sin\frac{(2r+1)\pi}{n},$$

from which expression we shall get the n different values of x, by putting $r=0$, 1, 2, &c. to $\frac{1}{2}n-1$ when n is even, or to $\frac{1}{2}(n-1)$ when n is odd, the last pair in the latter case being each $=-1$.

Hence, (i) if n be even,

$$(x^n+1)=(x^2-2x\cos\frac{\pi}{n}+1)\,(x^2-2x\cos\frac{3\pi}{n}+1)\ldots$$

or, (ii) if n be odd,

$$(x^n+1)=(x+1)\,(x^2-2x\cos\frac{\pi}{n}+1)\,(x^2-2x\cos\frac{3\pi}{n}+1)\ldots$$

where the factor $x+1$ arises from the root -1, obtained by putting $r=\frac{1}{2}(n-1)$.

25. If it be required to resolve x^n-a^n or x^n+a^n into factors, put $x=az$; then

$$x^n-a^n=a^n(z^n-1)=a^n(z^2-1)\,(z^2-2z\cos\frac{2\pi}{n}+1)\ldots \text{ if } n \text{ be even}$$

$$=(x^2-a^2)\,(x^2-2ax\cos\frac{2\pi}{n}+a^2)\ldots$$

and so with the other cases.

26. From the preceding results many remarkable factorials may be derived.

Since $x^{2n}-1=(x^2-1)\,(x^2-2x\cos\frac{\pi}{n}+1)\,(x^2-2x\cos\frac{2\pi}{n}+1)\ldots$

the index being even,

$$\therefore x^{2n-2}+x^{2n-4}+\&c+x^2+1=(x^2-2x\cos\frac{\pi}{n}+1)\,(x^2-2x\cos\frac{2\pi}{n}+1)\ldots$$

In this expression put $x = 1$; then we get

$$n = \left(2 - 2\cos\frac{\pi}{n}\right)\left(2 - 2\cos\frac{2\pi}{n}\right)\ldots\ldots\left(2 - 2\cos\frac{(n-1)\pi}{n}\right)$$

$$= 4\sin^2\frac{\pi}{2n} \,.\, 4\sin^2\frac{2\pi}{2n}\ldots\ldots 4\sin^2\frac{(n-1)\pi}{2n},$$

or $\sqrt{n} = 2^{n-1}\,.\,\sin\frac{\pi}{2n}\,.\,\sin\frac{2\pi}{2n}\ldots\ldots\sin\frac{(n-1)\pi}{2n}$, ... (1)

where we take the positive sign with the root, because each of the above angles being $< \frac{1}{2}\pi$, every sine will be positive.

So also

$$x^{2n+1} - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{2n+1} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{2n+1} + 1\right)\ldots$$

the index being odd;

∴ dividing by $x - 1$, and putting $x = 1$, we get

$$\sqrt{(2n+1)} = 2^n\sin\frac{\pi}{2n+1}\,.\,\sin\frac{2\pi}{2n+1}\ldots\ldots\sin\frac{n\pi}{2n+1} \;\ldots(2)$$

Again, $x^{2n} + 1 = \left(x^2 - 2x\cos\frac{\pi}{2n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{2n} + 1\right)\ldots$

put $x = 1$; then $2 = 4\sin^2\frac{\pi}{4n}\,.\,4\sin^2\frac{3\pi}{4n}\ldots\ldots 4\sin^2\frac{(2n-1)\pi}{4n}$,

$$\text{or } \sqrt{2} = 2^n\sin\frac{\pi}{4n}\,.\,\sin\frac{3\pi}{4n}\ldots\ldots\sin\frac{(2n-1)\pi}{4n}. \;\ldots..(3)$$

And $x^{2n+1} + 1 = (x+1)\left(x^2 - 2x\cos\frac{\pi}{2n+1} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{2n+1} + 1\right)$.

dividing by $x + 1$, and putting $x = 1$, we get

$$1 = 2^n\sin\frac{\pi}{2(2n+1)}\,.\,\sin\frac{3\pi}{2(2n+1)}\ldots\ldots\sin\frac{(2n-1)\pi}{2(2n+1)}\,.\;\ldots(4)$$

It is obvious that other results might be obtained in the sa way by putting $x = -1$, which would make the factors consist cosines instead of sines, and so, by division, we might arrive other similar results, where the factors are tangents.

Thus, in the equation which led to (1), putting $x = -1$, shall get $\sqrt{n} = 2^{n-1}\cos\frac{\pi}{2n}\,.\,\cos\frac{2\pi}{2n}\ldots\ldots\cos\frac{(n-1)\pi}{2n}$;

and therefore also, dividing (1) by this result, we have

$$1 = \tan\frac{\pi}{2n}\,.\,\tan\frac{2\pi}{2n}\ldots\ldots\tan\frac{(n-1)\pi}{2n}\,.\;\ldots\ldots\ldots(5)$$

27. *To resolve* $x^{2n} - 2x^n\cos\theta + 1$ *into quadratic facto*

Put $x^{2n} - 2x^n\cos\theta + 1 = 0$; then $x^{2n} - 2x^n\cos\theta + \cos^2\theta = -\sin$

hence $x^n = \cos\theta \pm \sqrt{-1}\sin\theta = \cos(2r\pi + \theta) \pm \sqrt{-1}\sin(2r\pi + \theta)$

$$\therefore\ x = \cos\frac{2r\pi+\theta}{n} \pm \sqrt{-1}\sin\frac{2r\pi+\theta}{n},$$

where the $2n$ different values of x may be obtained by taking r from 0 to $n-1$.

Hence, as in [23], we get

$$x^{2n} - 2x^n\cos\theta + 1 = \left(x^2 - 2x\cos\frac{\theta}{n} + 1\right)\left(x^2 - 2x\cos\frac{2\pi+\theta}{n} + 1\right)$$

$$\ldots\left(x^2 - 2x\cos\frac{2(n-1)\pi+\theta}{n} + 1\right);\ \ldots\ldots\ldots\ldots\ldots (1)$$

whence we get, putting $\frac{x}{a}$ for x, and multiplying each side by a^{2n},

$$x^{2n} - 2a^nx^n\cos\theta + a^{2n} = \left(x^2 - 2ax\cos\frac{\theta}{n} + a^2\right)\left(x^2 - 2ax\cos\frac{2\pi+\theta}{n} + a^2\right)\ldots$$

From the above result, putting $\theta = 0$, we get

$$x^{2n} - 2a^nx^n + a^{2n} = (x^n \sim a^n)^2 = (x^2 - 2ax + a^2)\left(x^2 - 2ax\cos\frac{2\pi}{n} + a^2\right)\ldots;$$

and, putting $\theta = \pi$,

$$x^{2n} + 2a^nx^n + a^{2n} = (x^n + a^n)^2 = \left(x^2 - 2ax\cos\frac{\pi}{n} + a^2\right)\left(x^2 - 2ax\cos\frac{3\pi}{n} + a^2\right)\ldots.$$

28. If we observe that

$$\cos\frac{2(n-1)\pi+\theta}{n} = \cos\left(2\pi - \frac{2\pi-\theta}{n}\right) = \cos\frac{2\pi-\theta}{n},\ \text{\&c.}$$

the expression (1) in the last article may be put into the form

$$x^{2n} - 2x^n\cos\theta + 1$$

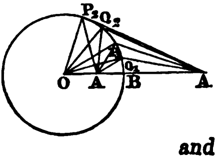
$$= \left(x^2 - 2x\cos\frac{\theta}{n} + 1\right)\left(x^2 - 2x\cos\frac{2\pi\pm\theta}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi\pm\theta}{n} + 1\right)\ldots,$$

where both signs are to be taken, and there must be altogether n quadratic factors—the last, consequently, being either a *single* or a *double* factor, according as n is even or odd.

COR. The above expression will be found to include those already found for $x^n \pm 1$; since, by putting $\theta = 0$ and $\theta = \pi$, we may get the resolution into factors of $(x^n \mp 1)^2$.

29. *Cotes's Properties of the Circle.*

Let A be any point in the radius OB, or in the radius OB produced, of a circle whose centre is O; and let the circumference be divided into n equal parts BP_1, P_1P_2, &c, and into $2n$ equal parts BQ_1, Q_1P_1, &c: then



$$AP_1.AP_2.AP_3\ldots\text{to } n \text{ factors} = OA^n \sim OB^n,$$

and $AQ_1.AQ_2.AQ_3\ldots$to n factors $= OA^n + OB^n$.

For let $OA = x$, $OB = a$; then we have

$$AP_1^2 = OA^2 - 2OA.OP_1\cos\frac{2\pi}{n} + OP_1^2 = x^2 - 2ax\cos\frac{2\pi}{n} + a^2, \text{ \&c}:$$

$$\therefore AP_1^2.AP_2^2.AP_3^2\ldots = \left(x^2 - 2ax\cos\frac{2\pi}{n} + a^2\right)\left(x^2 - 2ax\cos\frac{4\pi}{n} + a^2\right)\ldots$$

$$= (x^n \sim a^n)^2 \text{ by [27]};$$

or $AP_1.AP_2.AP_3\ldots = OA^n \sim OB^n = OB^n - OA^n$ or $OA^n - OB^n$, according as A is in OB or OB produced.

Again, $AQ_1.AP_1.AQ_2.AP_2\ldots\ldots = OA^{2n} \sim OB^{2n}$, by the above; and therefore, dividing this by the former result, we get

$$AQ_1.AQ_2.AQ_3\ldots n \text{ factors} = OA^n + OB^n.$$

30. *To resolve $\sin\theta$, $\cos\theta$, $\tan\theta$ into factors of sines, cosines, and tangents.*

In the expression (1) of [27], put $x = 1$: then

$$2 - 2\cos\theta = \left(2 - 2\cos\frac{\theta}{n}\right)\left(2 - 2\cos\frac{2\pi+\theta}{n}\right)\ldots\left(2 - 2\cos\frac{2(n-1)\pi+\theta}{n}\right),$$

$$\text{or } 4\sin^2\frac{\theta}{2} = 4\sin^2\frac{\theta}{2n}.4\sin^2\frac{2\pi+\theta}{2n}\ldots 4\sin^2\frac{2(n-1)\pi+\theta}{2n};$$

$$\text{whence } \sin\frac{\theta}{2} = 2^{n-1}\sin\frac{\theta}{2n}\sin\frac{2\pi+\theta}{2n}\sin\frac{4\pi+\theta}{2n}\ldots\sin\frac{2(n-1)\pi+\theta}{2n},$$

or writing 2θ for θ,

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\sin\frac{\pi+\theta}{n}\sin\frac{2\pi+\theta}{n}\ldots \text{ to } n \text{ factors}\ldots\ldots(1).$$

N.B. We have taken the sign of the square root (+), which, as will be shewn, is always the case in the above resolution of $\sin\theta$.

Again, in the expression (1) of [27], put $x = -1$;

$$\left.\begin{array}{ll}\text{then} & (n \text{ even})\ 2 - 2\cos\theta \\ \text{or} & (n \text{ odd})\ 2 + 2\cos\theta\end{array}\right\} = \left(2 + 2\cos\frac{\theta}{n}\right)\left(2 + 2\cos\frac{2\pi+\theta}{n}\right)\ldots;$$

whence, writing 2θ for θ, and proceeding as before, we get

$$\left.\begin{array}{l}(n \text{ even}) \pm\sin\theta \\ \text{or } (n \text{ odd}) \pm\cos\theta\end{array}\right\} = 2^{n-1}\cos\frac{\theta}{n}\cos\frac{\pi+\theta}{n}\cos\frac{2\pi+\theta}{n}\ldots\text{to } n \text{ factors}\ldots(2).$$

Dividing (1) by (2), we get

$$\left.\begin{array}{l}(n \text{ even}) \pm 1 \\ (n \text{ odd}) \pm\tan\theta\end{array}\right\} = \tan\frac{\theta}{n}\tan\frac{\pi+\theta}{n}\tan\frac{2\pi+\theta}{n}\ldots \text{ to } n \text{ factors}\ldots(3).$$

Cor. If in (1) above we write $\frac{1}{2}\pi$ for θ, we get

$$1 = 2^{n-1}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\sin\frac{5\pi}{2n}\ldots\text{to } n \text{ factors}:$$

if in (1) we write 0 for θ, we get

$$\sin\theta \div \sin\frac{\theta}{n} = n = 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\ \ldots \text{ to } (n-1) \text{ factors}:$$

if in (3) we write $\frac{1}{4}\pi$ for θ, we get, whether n be even or odd,

$$\pm 1 = \tan\frac{\pi}{4n}\tan\frac{5\pi}{4n}\tan\frac{9\pi}{4n}\ \ldots \text{ to } n \text{ factors.}$$

And many similar formulæ may easily be derived.

31. We shall now shew that in the expression (1) of [30], the (+) sign of the root is always to be taken.

Write $n\theta$ for θ in it: then

$$\sin n\theta = 2^{n-1}\sin\theta\sin(\theta + \frac{\pi}{n})\sin(\theta + \frac{2\pi}{n})\ldots\ldots \text{to } n \text{ factors.}$$

Now (i) let $n\theta = m\pi - a$, where $a < \pi$, and $m =$ zero or any positive integer: (this will include all *positive* values of θ, and all *negative* values between 0 and $-\pi$:) then $\sin n\theta = (-1)^{m-1}\sin a$, and will therefore have the sign of $(-1)^{m-1}$.

The general type of the factors on the right-hand side is

$$\sin(\theta + \frac{r\pi}{n}), \text{ where } r \text{ may range from 0 to } n-1:$$

now $\sin(\theta + \frac{r\pi}{n}) = \sin\frac{(m+r)\pi - a}{n}$, where, since $a < \pi$, the angle itself is always positive: hence, in the case of m positive, this factor is *positive* or *negative*, alternately, while $\frac{(m+r)\pi - a}{n}$ lies between 0 and π, π and 2π, 2π and 3π, &c, that is, it is *positive*, while $r < (n-m) + \frac{a}{\pi}$, or from $r = 0$ to $r = n - m$ inclusive, *negative* while $r > n - m < 2n - m + \frac{a}{\pi}$, or from $r = n - m + 1$ to $r = 2n - m$, inclusive, &c, and, generally, it has the sign of $(-1)^p$ from $r = pn - m + 1$ to $r = (p+1)n - m$ inclusive.

Hence (1) if $m < n$, we shall have $n - m + 1$ positive factors, (from $r = 0$ to $r = n - m$), and (there being n factors altogether) $m - 1$ negative; so that the sign of the product will be that of $(-1)^{m-1}$;

(2) if $m = n$, then we shall have only *one* positive factor (when $r = n - m = 0$), and $n - 1 = m - 1$ negative factors: so that here too the sign of the product will be that of $(-1)^{m-1}$;

(3) if $m > n$, or if $m = pn + m'$, suppose, (where $m' < n$,) then the lowest value of r $(r = 0)$ will occur between $pn - m + 1$ and $(p + 1)\,n - m$, namely when $r = (pn - m) + m' = 0$; hence there will be $\{(p + 1)\,n - m\} - \{(pn - m) + m'\} + 1$ or $n - m' + 1$ factors with the sign $(-1)^p$, and the remaining $m' - 1$ factors with the sign $(-1)^{p+1}$: so that the sign of the product will be that of

$$(-1)^{(n-m'+1)\,p+(m'-1)(p+1)} \text{ or } (-1)^{np+m'-1} = (-1)^{m-1}, \text{ as before.}$$

Hence, in all cases, when m is *positive*, the sign of the product of the n factors is that of $(-1)^{m-1}$, which is the same as that of $\sin n\theta$.

(ii) Let $n\theta = -m\pi - a$; (this will include all *negative* values of θ, from $-\pi$ downwards:) then $\sin n\theta = (-1)^{m-1} \sin a$, as before; and $\sin(\theta + \frac{r\pi}{n}) = \sin\frac{(r - m)\,\pi - a}{n}$, where, since $a < \pi$, the angle is positive, from $r = m + 1$ to $r = n - 1$, (its highest value,) and negative for all values of r less than $m + 1$. Now, while the angle is positive, its sine is positive while $\frac{(r - m)\,\pi - a}{n}$ lies between 0 and π, that is, from $r = m + 1$ to $r = m + n$, both inclusive; and hence *all* the factors will be positive, for which the angle is positive, namely, from $r = m + 1$ to $r = n - 1$. But, when the angle is negative, its sine is *negative* or *positive*, alternately, while $\frac{(r - m)\,\pi - a}{n}$ lies between 0 and $-\pi$, $-\pi$ and -2π, &c, that is, it is *positive* from $r = m$ to $r = m - n + 1$, *negative* from $r = m - n$ to $r = m - 2n + 1$, &c; and, generally, it has the sign of $(-1)^p$ from $r = m - (p - 1)\,n$ to $r = m - pn + 1$.

Hence, (1) if $m < n$, we shall have $n - m - 1$ factors positive, (from $r = m + 1$ to $r = n - 1$,) and, consequently, $m + 1$ negative factors, so that the sign of the product will be that of $(-1)^{m+1} = (-1)^{m-1}$:

(2) if $m = n$, there will be n negative factors (from $r = m - 1$ or $n - 1$ (its highest value) to $r = m - n + 1 = 1$,) and *one* positive factor (when $r = m - n = 0$); so that here too the sign of the product will be $(-1)^{m-1}$:

(3) if $m > n$, or if $m = pn + m'$, where $m' < n$, then the lowest value of r $(r = 0)$ will occur between $r = m - pn$, and $r = m - (p + 1)\,n + 1$, and will be when $r = m - pn - m' = 0$; hence there will be $(m - pn) - (m - pn - m') + 1 = m' + 1$ factors with the sign $(-1)^{p+1}$

and the remaining $n - m' - 1$ with the sign $(-1)^p$; so that the sign of the product will be that of

$$(-1)^{(m'+1)(p+1)+(n-m'-1)p} = (-1)^{m'+np+1} = (-1)^{m+1} \text{ or } (-1)^{m-1}, \text{ as before:}$$

that is, in all cases, the product of the n factors will have the same sign as $\sin n\theta$.

Similar reasonings would enable us to determine in what cases the + or − sign is to be used in the resolution of $\cos\theta$ and $\tan\theta$. For example, in [30 Cor.], if $n = 1$, then $\tan\frac{1}{4}\pi = +1$, if $n = 2$, then $\tan\frac{1}{8}\pi \tan\frac{5}{8}\pi = -1$, if $n = 3$, then $\tan\frac{1}{12}\pi \tan\frac{5}{12}\pi \tan\frac{9}{12}\pi = -1$, two tangents in the last case being positive, the other negative.

32. *To resolve* $\sin n\theta$ *and* $\cos n\theta$ *into quadratic factors.*

The expression (1) in [30] for $\sin\theta$, (observing that

$$\sin\frac{(n-1)\pi + \theta}{n} = \sin(\pi - \frac{\pi - \theta}{n}) = \sin\frac{\pi - \theta}{n},)$$

may be put into the form

$$\sin\theta = 2^{n-1} \sin\frac{\theta}{n} \sin\frac{\pi \pm \theta}{n} \sin\frac{2\pi \pm \theta}{n} \ldots,$$

where both signs are to be taken, and there are to be n factors altogether, the last being *single* (with the upper sign only) or *double*, according as n is even or odd.

Write $n\theta$ for θ, and $2na$ for π, in the above expression.

(1) If n be *odd*,

$$\sin n\theta = 2^{n-1} \sin\theta \sin(2a \pm \theta) \sin(4a \pm \theta) \ldots \sin\{(n-1)a \pm \theta\};$$

whence, writing $\frac{1}{2}\pi + n\theta$ for $n\theta$, or $a + \theta$ for θ, we get also

$$\cos n\theta = 2^{n-1} \sin(a + \theta) \sin(a - \theta) \sin(3a + \theta) \sin(3a - \theta) \ldots$$
$$\sin\{(n-2)a - \theta\} \sin(na + \theta),$$
$$= 2^{n-1} \cos\theta \sin(a \pm \theta) \sin(3a \pm \theta) \ldots \sin\{(n-2)a \pm \theta\},$$

since the last factor $\sin(na + \theta) = \sin(\frac{1}{2}\pi + \theta) = \cos\theta$.

Hence, observing that $\sin(a+\theta)\sin(a-\theta) = \sin^2 a - \sin^2\theta$, we get

$$\sin n\theta = 2^{n-1} \sin\theta (\sin^2 2a - \sin^2\theta)(\sin^2 4a - \sin^2\theta) \ldots \{\sin^2(n-1)a - \sin^2\theta\},$$
$$\cos n\theta = 2^{n-1} \cos\theta (\sin^2 a - \sin^2\theta)(\sin^2 3a - \sin^2\theta) \ldots \{\sin^2(n-2)a - \sin^2\theta\}.$$

(2) If n be *even*, we have, as before,

$$\sin n\theta = 2^{n-1} \sin\theta \sin(2a \pm \theta) \sin(4a \pm \theta) \ldots \{\sin(n-2)a \pm \theta\} \sin(na + \theta)$$
$$= 2^{n-1} \sin\theta \cos\theta (\sin^2 2a - \sin^2\theta)(\sin^2 4a - \sin^2\theta) \ldots \{\sin^2(n-2)a - \sin^2\theta\};$$
$$\cos n\theta = 2^{n-1} \sin(a + \theta) \sin(a - \theta) \sin(3a + \theta) \sin(3a - \theta) \ldots$$
$$\sin\{(n-1)a + \theta\} \sin\{(n+1)a + \theta\}$$
$$= 2^{n-1}(\sin^2 a - \sin^2\theta)(\sin^2 3a - \sin^2\theta) \ldots \{\sin^2(n-1)a - \sin^2\theta\},$$

since $\sin\{(n+1)a+\theta\} = \sin(\frac{1}{2}\pi + a + \theta\} = \sin(\frac{1}{2}\pi - a - \theta) = \sin\{(n-1)a - \theta\}$.

33. We may employ the preceding results to resolve $\sin\theta$ an $\cos\theta$ into quadratic factors in terms of θ.

Thus in the formulæ for $\sin n\theta$, $\cos n\theta$, n odd, write θ for $n\theta$: the

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\left(\sin^2 2a - \sin^2\frac{\theta}{n}\right)\left(\sin^2 4a - \sin^2\frac{\theta}{n}\right)\dots$$

$$= 2^{n-1}\sin^2 2a\sin^2 4a\dots\sin\frac{\theta}{n}\left(1-\operatorname{cosec}^2 2a\sin^2\frac{\theta}{n}\right)\left(1-\operatorname{cosec}^2 4a\sin^2\frac{\theta}{n}\right)$$

To determine the value of the constant factor, put $\theta = 0$:

then we have $\sin\theta \div \sin\dfrac{\theta}{n} = n = 2^{n-1}\sin^2 2a\sin^2 4a\dots;$

and $\therefore$ $\sin\theta = n\sin\dfrac{\theta}{n}\left(1-\operatorname{cosec}^2 2a\sin^2\dfrac{\theta}{n}\right)\left(1-\operatorname{cosec}^2 4a\sin^2\dfrac{\theta}{n}\right).$

Put $n = \infty$, then $n\sin\dfrac{\theta}{n} = \theta$, $\sin\dfrac{\theta}{n}\div\sin 2a = \dfrac{\theta}{n}\div\dfrac{\pi}{n} = \dfrac{\theta}{\pi}$, &c

and, consequently, $\sin\theta = \theta\left(1-\dfrac{\theta^2}{\pi^2}\right)\left(1-\dfrac{\theta^2}{2^2\pi^2}\right)\left(1-\dfrac{\theta^2}{3^2\pi^2}\right)\dots$

So $\cos\theta = 2^{n-1}\sin^2 a\sin^2 3a\dots\cos\dfrac{\theta}{n}\left(1-\operatorname{cosec}^2 a\sin^2\dfrac{\theta}{n}\right)\dots;$

put $\theta = 0$; then $1 = 2^{n-1}\sin^2 a\sin^2 3a\dots,$

and $\therefore$ $\cos\theta = \cos\dfrac{\theta}{n}\left(1-\operatorname{cosec}^2 a\sin^2\dfrac{\theta}{n}\right)\left(1-\operatorname{cosec}^2 3a\sin^2\dfrac{\theta}{n}\right)\dots$

$$= \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\dots, \text{ when } n = \infty.$$

But the above resolutions may be effected more directly as follo

34. *To resolve* sin θ *into a product of quadratic facto in terms of* θ.

Since the values of θ, which make $\sin\theta = 0$, are 0, $\pm\pi$, ± 2 $\pm 3\pi$, &c, therefore, by the Theory of Equations, we have

$$\sin\theta = a\theta(\theta^2-\pi^2)(\theta^2-2^2\pi^2)(\theta^2-3^2\pi^2)\dots,$$

where a is some constant factor, not known at present: hence

$$\frac{\sin\theta}{\theta} = \pm a\pi^2.2^2\pi^2.3^2\pi^2\dots\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\dots$$

Put $\theta = 0$; then $1 = \pm a\pi^2.2^2\pi^2.3^2\pi^2\dots$, which determines th constant quantity;

$$\therefore \sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\dots$$

35. *To resolve* $\cos\theta$ *into a product of quadratic factors in terms of* θ.

The values of θ which make $\cos\theta = 0$ are $\pm\frac{1}{2}\pi$, $\pm\frac{3}{2}\pi$, &c;

$$\therefore \cos\theta = b\left(\theta^2 - \frac{\pi^2}{4}\right)\left(\theta^2 - \frac{3^2\pi^2}{4}\right)\left(\theta^2 - \frac{5^2\pi^2}{4}\right)\ldots$$

$$= \pm b \,.\, \frac{\pi^2}{4} \,.\, \frac{3^2\pi^2}{4} \,.\, \frac{5^2\pi^2}{4} \ldots \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\ldots$$

Put $\theta = 0$; then $1 = \pm b \,.\, \frac{1}{4}\pi^2 \,.\, \frac{1}{4}(3^2\pi^2) \,.\, \frac{1}{4}(5^2\pi^2)\ldots$;

$$\therefore \cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\ldots$$

36. In the expression just found for $\sin\theta$, put $\theta = \frac{1}{2}\pi$: then

$$1 = \frac{\pi}{2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\ldots = \frac{\pi}{2} \,.\, \frac{1.3}{2^2} \,.\, \frac{3.5}{4^2} \,.\, \frac{5.7}{6^2} \ldots;$$

$$\therefore \frac{\pi}{2} = \frac{2^2}{1.3} \,.\, \frac{4^2}{3.5} \,.\, \frac{6^2}{5.7} \ldots,$$

which is known as *Wallis's* expression for π.

37. If we expand the expressions in [34], and compare the coefficients of the powers of θ with those in the common formulæ

$$\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \&c, \quad \cos\theta = 1 - \frac{\theta^2}{1.2} + \&c,$$

we get $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c = \frac{\pi^2}{6}$, $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c = \frac{\pi^2}{8}$,

with other similar results.

38. *To resolve* $e^x + e^{-x}$ *into quadratic factors.*

In the factorials, [34] and [35], for $\sin\theta$ and $\cos\theta$, use the exponential expressions found in [7]:

$$\text{then } e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}} = 2\sqrt{-1}\,\theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\ldots$$

$$e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}} = 2\left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\ldots$$

Write x for $\theta\sqrt{-1}$: then

$$e^x - e^{-x} = 2x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\ldots$$

$$e^x + e^{-x} = 2\left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4n^2}{3^2\pi^2}\right)\ldots$$

CHAPTER IV.

ON THE SUMMATION OF TRIGONOMETRICAL SERIES.

39. *To find the sums of the sines and cosines of a series of angles in* A. P.

We have $\cos(\theta - \frac{1}{2}a) - \cos(\theta + \frac{1}{2}a) = 2\sin\frac{1}{2}a \sin\theta$,

$$\cos(\theta + \tfrac{1}{2}a) - \cos(\theta + \tfrac{3}{2}a) = 2\sin\tfrac{1}{2}a \sin(\theta + a), \ \&c:$$

hence, by addition,

$$\cos(\theta - \tfrac{1}{2}a) - \cos\{\theta + \tfrac{1}{2}(2n-1)a\} = 2\sin\tfrac{1}{2}a\{\sin\theta + \sin(\theta + a) + \&c\};$$

$$\therefore S = \frac{\cos(\theta - \frac{1}{2}a) - \cos\{\theta + \frac{1}{2}(2n-1)a\}}{2\sin\frac{1}{2}a} = \frac{\sin\{\theta + \frac{1}{2}(n-1)a\}\sin\frac{1}{2}na}{\sin\frac{1}{2}a}.$$

Similarly, we may sum $\cos\theta + \cos(\theta + a) + \&c$; or, if we write $\frac{1}{2}\pi + \theta$ for θ in the above, we get at once

$$S' = \frac{\cos\{\theta + \frac{1}{2}(n-1)a\}\sin\frac{1}{2}na}{\sin\frac{1}{2}a}.$$

COR. Putting $a = \theta$ in the above results, we have

$$\sin\theta + \sin 2\theta + \&c + \sin n\theta = \frac{\sin\frac{1}{2}(n+1)\theta \sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta},$$

$$\cos\theta + \cos 2\theta + \&c + \cos n\theta = \frac{\cos\frac{1}{2}(n+1)\theta \sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}.$$

40. The sums of the squares, cubes, &c, of sines and cosines of angles in A. P., may be found by changing the series into one involving the sines and cosines of multiple angles.

Thus, to sum $\cos^3\theta + \cos^3(\theta + a) + \&c.$ to n terms:

$$\text{here by (75) } \cos^3\theta = \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos 3\theta;$$

so that, by [39], we get

$$S = \tfrac{3}{4}\{\cos\theta + \cos(\theta + a) + \&c\} + \tfrac{1}{4}\{\cos 3\theta + \cos(3\theta + 3a) + \&c\}$$

$$= \frac{3}{4}\,\frac{\cos\{\theta + \frac{1}{2}(n-1)a\}\sin\frac{1}{2}na}{\sin\frac{1}{2}a} + \frac{1}{4}\,\frac{\cos\{3\theta + \frac{1}{2}(n-1)3a\}\sin\frac{3}{2}na}{\sin\frac{3}{2}a}.$$

41. *To sum* $\sin\theta - \sin(\theta + a) + \sin(\theta + 2a) - \&c,$

$$\cos\theta - \cos(\theta + a) - \cos(\theta + 2a) - \&c.$$

Here $\sin(\theta + \frac{1}{2}a) + \sin(\theta - \frac{1}{2}a) = 2\cos\frac{1}{2}a \sin\theta$

$$-\sin(\theta + \tfrac{3}{2}a) - \sin(\theta + \tfrac{1}{2}a) = -2\cos\tfrac{1}{2}a \sin(\theta + a), \ \&c:$$

hence $\pm\sin\{\theta + \frac{1}{2}(2n-1)a\} + \sin(\theta - \frac{1}{2}a) = 2\cos\frac{1}{2}a\{\sin\theta - \sin(\theta + a) + \&c\}$,

$+$ or $-$ according as n is odd or even;

$$\therefore\ S = \frac{\sin(\theta - \frac{1}{2}a) \pm \sin\{\theta + \frac{1}{2}(2n-1)a\}}{2\cos\frac{1}{2}a}$$

$$= \frac{\sin\{\theta + \frac{1}{2}(n-1)a\}\cos\frac{1}{2}na}{\cos\frac{1}{2}a},\ \text{or}\ -\frac{\cos\{\theta + \frac{1}{2}(n-1)a\}\sin\frac{1}{2}na}{\cos\frac{1}{2}a},$$

according as n is *odd* or *even*.

By similar reasoning, or, at once, by writing $\frac{1}{2}\pi + \theta$ for θ in the above, we may get the sum of $\cos\theta - \cos(\theta + a) + \&c$; and then, by putting $a = \theta$, we may obtain the sums of

$$\sin\theta - \sin 2\theta + \sin 3\theta - \&c, \quad \cos\theta - \cos 2\theta + \cos 3\theta - \&c.$$

42. ***To sum*** $\sin\theta + x\sin(\theta + a) + x^2\sin(\theta + 2a) + \&c,$

$$\cos\theta + x\cos(\theta + a) + x^2\cos(\theta + 2a) + \&c.$$

Here $x\{\sin(\theta + a) + \sin(\theta - a)\} = 2x\sin\theta\cos a,$

$$x^2\{\sin(\theta + 2a) + \sin\theta\} = 2x^2\sin(\theta + a)\cos a,$$

$$x^3\{\sin(\theta + 3a) + \sin(\theta + a)\} = 2x^3\sin(\theta + 2a)\cos a,\ \&c,$$

$$\therefore\ \{S - \sin\theta + x^n\sin(\theta + na)\}$$

$$+ [x\sin(\theta - a) + x^2S - x^{n+1}\sin\{\theta + (n-1)a\}] = 2xS\cos a,$$

whence

$$S = \frac{\sin\theta - x\sin(\theta - a) - x^n\sin(\theta + na) + x^{n+1}\sin\{\theta + (n-1)a\}}{1 - 2x\cos a + x^2}.$$

By similar reasoning, or, at once, writing $\frac{1}{2}\pi + \theta$ for θ, we get

$$\cos\theta + x\cos(\theta + a) + x^2\cos(\theta + 2a) + \&c\ \text{to}\ n\ \text{terms}$$

$$= \frac{\cos\theta - x\cos(\theta - a) - x^n\cos(\theta + na) + x^{n+1}\cos\{\theta + (n-1)a\}}{1 - 2x\cos a + x^2}.$$

These results (by writing $-x$ for x) include the sums of

$$\sin\theta - x\sin(\theta + a) + \&c, \quad \cos\theta - x\cos(\theta + a) + \&c:$$

hence, putting $x = \pm 1$, we may get the sums already found, of
$\sin\theta \pm \sin(\theta + a) + \sin(\theta + 2a) \pm \&c, \cos\theta \pm \cos(\theta + a) + \cos(\theta + 2a) \pm \&c;$
or, again, putting $a = \theta$, we may get the sums of

$$\sin\theta \pm x\sin 2\theta + x^2\sin 3\theta \pm \&c, \quad \cos\theta \pm x\cos 2\theta + x^2\cos 3\theta \pm \&c.$$

Cor. If $x < 1$, then, when n is very great, x^n and x^{n+1} vanish;

and $\quad \sin\theta \pm x\sin(\theta + a) + \&c$ *ad inf.* $= \dfrac{\sin\theta \mp x\sin(\theta - a)}{1 \mp 2x\cos a + x^2},$

$$\cos\theta \pm x\cos(\theta + a) + \&c\ \textit{ad inf.} = \frac{\cos\theta \mp x\cos(\theta - a)}{1 \mp 2x\cos a + x^2}:$$

or, putting $a = \theta$,

$$\sin\theta \pm x\sin 2\theta + \&c\ \textit{ad inf.} = \frac{\sin\theta}{1 \mp 2x\cos\theta + x^2},$$

$$\cos\theta \pm x\cos 2\theta + \&c\ \textit{ad inf.} = \frac{\cos\theta \mp 1}{1 \mp 2x\cos\theta + x^2}.$$

43. We shall now illustrate, by a few examples, the use of th results in [13, 14].

Ex. 1. *To sum* $x \sin\theta + x^2 \sin 2\theta + x^3 \sin 3\theta +$ &c *to* n *terms.*

Put $2\sqrt{-1}\sin\theta = z - \frac{1}{z}$; then we have

$$2\sqrt{-1}\,S = \left(xz - \frac{x}{z}\right) + \left(x^2z^2 - \frac{x^2}{z^2}\right) + \&c = (xz + x^2z^2 + \&c) - \left(\frac{x}{z} + \frac{x^2}{z^2} + \&\right.$$

$$= \frac{(xz)^{n+1} - xz}{xz - 1} - \frac{(xz^{-1})^{n+1} - xz^{-1}}{xz^{-1} - 1}$$

$$= \frac{x^{n+2}(z^n - z^{-n}) - x^{n+1}(z^{n+1} - z^{-(n+1)}) + x(z - z^{-1})}{x^2 - x(z + z^{-1}) + 1};$$

or $$S = \frac{x^{n+2}\sin n\theta - x^{n+1}\sin(n+1)\theta + x\sin\theta}{x^2 - 2x\cos\theta + 1},$$

the same result as we should obtain in [42], by putting $a = \theta$, a multiplying by x.

Ex. 2. *To sum ad inf.* $(x < 1)$

$$x\cos(\theta + a) + \frac{x^2}{1.2}\cos(\theta + 2a) + \frac{x^3}{1.2.3}\cos(\theta + 3a) + \&c.$$

Let $2\cos\theta = y + y^{-1} = e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}$, $\quad 2\cos a = z + z^{-1} = e^{a\sqrt{-1}} + e^{-a\sqrt{}}$

then $$2\Sigma = x(yz + y^{-1}z^{-1}) + \frac{x^2}{1.2}(yz^2 + y^{-1}z^{-2})\ \&c$$

$$= y\left(xz + \frac{x^2z^2}{1.2} + \&c\right) + y^{-1}\left(xz^{-1} + \frac{x^2z^{-2}}{1.2} + \&c\right)$$

$$= y(e^{xz} - 1) + y^{-1}(e^{xz^{-1}} - 1)$$

$$= e^{\theta\sqrt{-1}} \times e^{x(\cos a + \sqrt{-1}\sin a)} + e^{-\theta\sqrt{-1}} \times e^{x(\cos a - \sqrt{-1}\sin a)} - (y + y^{-1})$$

$$= e^{x\cos a}\left\{e^{(\theta + x\sin a)\sqrt{-1}} + e^{-(\theta + x\sin a)\sqrt{-1}}\right\} - (y + y^{-1})$$

$$= e^{x\cos a} \times 2\cos(\theta + x\sin a) - 2\cos\theta:$$

hence $S = e^{x\cos a}\cos(\theta + x\sin a) - \cos\theta$.

Writing $\frac{1}{2}\pi + \theta$ for θ, we get

$$x\sin(\theta + a) + \frac{x^2}{1.2}\sin(\theta + 2a) + \&c = \sin\theta - e^{x\cos a}\sin(\theta + x\sin$$

Cor. Putting $\theta = 0$ and $a = \theta$, in these two results, we get

$$x\cos\theta + \frac{x^2}{1.2}\cos 2\theta + \&c = e^{x\cos\theta}\cos(x\sin\theta) - 1,$$

$$x\sin\theta + \frac{x^2}{1.2}\sin 2\theta + \&c = -e^{x\cos\theta}\sin(x\sin\theta).$$

Ex. 3. *To sum ad inf.* $(x < 1)$

$$x\cos(\theta + a) + \frac{x(x-1)}{1.2}\cos(\theta + 2a) + \frac{x(x-1)(x-2)}{1.2.3}\cos(\theta + 3a) +$$

Reasoning as in Ex. 2, we get

$$2\Sigma = x\,(yz + y^{-1}z^{-1}) + \frac{x\,(x-1)}{1.2}\,(yz^2 + y^{-1}z^{-2}) + \&c$$
$$= y\,\{xz + \frac{x\,(x-1)}{1.2}\,z^2 + \&c\} + y^{-1}\,\{xz^{-1} + \frac{x\,(x-1)}{1.2}\,z^{-2} + \&c\}$$
$$= y\,\{(1+z)^x - 1\} + y^{-1}\{(1+z^{-1})^x - 1\}$$
$$= yz^{\frac{1}{2}x}\,(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^x + y^{-1}z^{-\frac{1}{2}x}\,(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^x - (y + y^{-1})$$
$$= (yz^{\frac{1}{2}x} + y^{-1}z^{-\frac{1}{2}x})\,(z^{\frac{1}{2}} + z^{-\frac{1}{2}})^x - (y + y^{-1})$$
$$= 2\cos(\theta + \tfrac{1}{2}xa)\,(2\cos\tfrac{1}{2}a)^x - 2\cos\theta:$$
$$\therefore\ \Sigma = 2^x\cos(\theta + \tfrac{1}{2}xa)\,(\cos\tfrac{1}{2}a)^x - \cos\theta.$$

Writing $\frac{1}{2}\pi + \theta$ for θ, we get

$$x\sin(\theta+a) + \frac{x(x-1)}{1.2}\sin(\theta+2a) + \&c = 2^x\sin(\theta+\tfrac{1}{2}xa)\,(\cos\tfrac{1}{2}a)^x - \sin\theta.$$

Cor. Putting $\theta = 0$ and $a = \theta$, in these two results, we get

$$x\cos\theta + \frac{x\,(x-1)}{1.2}\cos 2\theta + \&c = 2^x\cos\tfrac{1}{2}x\theta\,(\cos\tfrac{1}{2}\theta)^x - 1,$$
$$x\sin\theta + \frac{x\,(x-1)}{1.2}\sin 2\theta + \&c = 2^x\sin\tfrac{1}{2}x\theta\,(\cos\tfrac{1}{2}\theta)^x.$$

44. *To sum* $\sec\theta\sec 2\theta + \sec 2\theta\sec 3\theta + \&c$ *to* n *terms.*

We have $\tan(n+1)\,\theta - \tan n\theta = \sin\theta\sec n\theta\sec(n+1)\,\theta$;
hence, putting for n the values 1, 2, 3, &c, successively,

$$\sin\theta\,.\,\sec\theta\sec 2\theta = \tan 2\theta - \tan\theta,$$
$$\sin\theta\,.\sec 2\theta\sec 3\theta = \tan 3\theta - \tan 2\theta,\ \&c;$$
$$\therefore\ \sin\theta\,.\,S = \tan(n+1)\,\theta - \tan\theta,\ \text{ or }\ S = \operatorname{cosec}\theta\,\{\tan(n+1)\,\theta - \tan\theta\}.$$

45. *To sum* $\operatorname{cosec}\theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta + \&c$ *to* n *terms.*

We have $\operatorname{cosec}\theta + \cot\theta = = \cot\tfrac{1}{2}\theta$;
hence

$$\operatorname{cosec}\theta = \cot\tfrac{1}{2}\theta - \cot\theta,$$
$$\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta,\ \&c;$$
$$\therefore\ S\ (\text{to } n \text{ terms}) = \cot\tfrac{1}{2}\theta - \cot 2^{n-1}\theta.$$

46. *To sum* $\frac{1}{2}\tan\frac{1}{2}\theta + \frac{1}{4}\tan\frac{1}{4}\theta + \frac{1}{8}\tan\frac{1}{8}\theta + \&c.$

We have $\cot\tfrac{1}{2}\theta - \tan\tfrac{1}{2}\theta = 2\cot\theta$;
hence

$$\tfrac{1}{2}\tan\tfrac{1}{2}\theta = \tfrac{1}{2}\cot\tfrac{1}{2}\theta - \cot\theta,$$
$$\tfrac{1}{4}\tan\tfrac{1}{4}\theta = \tfrac{1}{4}\cot\tfrac{1}{4}\theta - \tfrac{1}{2}\cot\tfrac{1}{2}\theta,\ \&c;$$
$$\therefore\ S\ (\text{to } n \text{ terms}) = \frac{1}{2^n}\cot\frac{\theta}{2^n} - \cot\theta:$$

$$\text{and }\ \Sigma\,(ad\ inf.) = \frac{1}{\theta}\left(\frac{\theta}{2^n} \div \tan\frac{\theta}{2^n}\right) - \cot\theta = \frac{1}{\theta} - \cot\theta.$$

47. *To sum*

$$(\tan\theta+\cot\theta)+(\tan 2\theta+\cot 2\theta)+(\tan 4\theta+\cot 4\theta)+$$

As in [46], $\tan\theta+\cot\theta=2\cot\theta-2\cot 2\theta,$

$$\tan 2\theta+\cot 2\theta=2\cot 2\theta-2\cot 4\theta, \text{ \&c}:$$

$\therefore S$ (to n terms) $=2\cot\theta-2\cot 2^n\theta.$

48. It will be sufficient to suggest to the Student, wh acquainted with the processes of the Differential and Int Calculus, that many series may be summed by differentiatin integrating other series or expressions, found as above.

Thus, [31] $\sin n\theta=2^{n-1}\sin\theta\sin(\theta+\frac{\pi}{n})\sin(\theta+\frac{2\pi}{n})\ldots;$

therefore, taking the logs and differentiating, we have

$$n\cot n\theta=\cot\theta+\cot(\theta+\frac{\pi}{n})+\cot(\theta+\frac{2\pi}{n})+\text{\&c};$$

and, differentiating again,

$$n^2\operatorname{cosec}^2 n\theta=\operatorname{cosec}^2\theta+\operatorname{cosec}^2(\theta+\frac{\pi}{n})+\operatorname{cosec}^2(\theta+\frac{2\pi}{n})+\text{\&c}$$

Again, since $\sin\theta=\theta(1-\frac{\theta^2}{\pi^2})(1-\frac{\theta^2}{2^2\pi^2})(1-\frac{\theta^2}{3^2\pi^2})\ldots,$

we get, in the same way,

$$\cot\theta=\frac{1}{\theta}-\frac{2\theta}{\pi^2-\theta^2}-\frac{2\theta}{2^2\pi^2-\theta^2}-\text{\&c.}$$

So also since, [46], writing 2θ for θ, and multiplying by 2, we

$$\tan\theta+\tfrac{1}{2}\tan\tfrac{1}{2}\theta+\tfrac{1}{4}\tan\tfrac{1}{4}\theta+\text{\&c } \textit{ad inf.}=\frac{1}{\theta}-2\cot 2\theta,$$

we get, by differentiation,

$$\sec^2\theta+\tfrac{1}{4}\sec^2\tfrac{1}{2}\theta+\tfrac{1}{16}\sec^2\tfrac{1}{4}\theta+\text{\&c } \textit{ad inf.}=-\frac{1}{\theta^2}+4\operatorname{cosec}^2 2$$

49. *Given* $\sin\theta=m\sin(\theta+a)$: *to expand* θ *in a s ascending by powers of* m.

Here $\sin\theta=m(\sin\theta\cos a+\cos\theta\sin a)$, or $\tan\theta=\frac{m\sin}{1-m\,c}$

putting $2\cos\theta=y+y^{-1}$ and $2\cos a=x+x^{-1}$, we get

$$\frac{y-y^{-1}}{y+y^{-1}}=\frac{m(x-x^{-1})}{2-m(x+x^{-1})},\quad\text{or }\frac{y}{y^{-1}}=y^2=e^{2\theta\sqrt{-1}}=\frac{1-mx^{-1}}{1-mx};$$

taking logs, $2\theta\sqrt{-1}=-(mx^{-1}+\tfrac{1}{2}m^2x^{-2}+\text{\&c})+(mx+\tfrac{1}{2}m^2x^2+$

$$=m(x-x^{-1})+\tfrac{1}{2}m^2(x^2-x^{-2})+\text{\&c},$$

$$\text{or }\theta=m\sin a+\tfrac{1}{2}m^2\sin 2a+\tfrac{1}{3}m^3\sin 3a+\text{\&c.}$$

The equation given above may also be put into the form

$$\frac{\sin(\theta+a)+\sin\theta}{\sin(\theta+a)-\sin\theta}=\frac{1+m}{1-m}=\frac{\tan(\theta+\frac{1}{2}a)}{\tan\frac{1}{2}a},\ \text{or}\ \tan(\theta+\tfrac{1}{2}a)=\frac{1+m}{1-m}\tan\tfrac{1}{2}a,$$

which, of course, would lead to the same result for θ in terms of a.

Since, in any triangle, $\sin A=\frac{a}{b}\sin(A+C)$, in the above write $\frac{a}{b}$ for m and C for a, and let θ represent the circular measure of A'';

then θ or (95) $A\sin 1''=\frac{a}{b}\sin C+\frac{a^2}{2b^2}\sin 2C+\frac{a^3}{3b^3}\sin 3C+\&c$,

a series for expressing one angle of a triangle in terms of another and the two sides containing the latter.

50. *To shew that, in any plane triangle,*

$$\log_e c=\log_e a-\{\frac{b}{a}\cos C+\frac{b^2}{2a^2}\cos 2C+\frac{b^3}{3a^3}\cos 3C+\&c\}.$$

We have $c^2=a^2+b^2-2ab\cos C=a^2+b^2-ab(x+x^{-1})$, if $2\cos C=x+x^{-1}$;

$$\therefore\ c^2=(a-bx)(a-bx^{-1}),\ \text{and}\ \frac{c^2}{a^2}=(1-\frac{b}{a}x)(1-\frac{b}{a}x^{-1}):$$

$$\therefore 2\log_e\frac{c}{a}=\log_e(1-\frac{b}{a}x)+\log_e(1-\frac{b}{a}x^{-1})=-\frac{b}{a}(x+x^{-1})-\frac{b^2}{2a^2}(x^2+x^{-2})-\&c;$$

or $\quad\log_e c=\log_e a-\{\frac{b}{a}\cos C+\frac{b^2}{2a^2}\cos 2C+\frac{b^3}{3a^3}\cos 3C+\&c\}$.

COR. Put $a^2+b^2=1$, $2ab=p$, and $C=a$, so that $c^2=1-p\cos a$; then $a\pm b=\sqrt{(1\pm p)}$, or $a=\frac{1}{2}\{\sqrt{(1+p)}+\sqrt{(1-p)}\}$, $b=\frac{1}{2}\{\sqrt{(1+p)}-\sqrt{(1-p)}\}$:

$$\therefore\ \log_e(1-p\cos a)=2\log_e c=2\{\log_e a-(n\cos a+\tfrac{1}{2}n^2\cos 2a+\&c)\},$$

$$\text{where}\ n=\frac{b}{a}=\frac{\sqrt{(1+p)}-\sqrt{(1-p)}}{\sqrt{(1+p)}+\sqrt{(1-p)}}=\frac{1-\sqrt{(1-p^2)}}{p}.$$

51. *To expand* $(a^2-2ab\cos a+b^2)^{-m}$ *in a series of the form* $A_0+A_1\cos a+A_2\cos 2a+\&c$.

We have $a^2-2ab\cos a+b^2)^{-m}$

$$=\{a^2-ab(x+x^{-1})+b^2\}^{-m},\ (\text{if}\ 2\cos a=x+x^{-1},)$$

$$=(a-bx)^{-m}(a-bx^{-1})^{-m}=a^{-2m}(1-nx)^{-m}(1-nx^{-1})^{-m},\ (\text{if}\ n=\frac{b}{a},)$$

$$=a^{-2m}(c_0+c_1x+c_2x^2+\&c)\times(c_0+c_1x^{-1}+c_2x^{-2}+\&c),$$

(where c_0, c_1, &c are the successive terms of the expansion of $(1-n)^{-m}$,

$$=a^{-2m}\{(c_0^2+c_1^2+c_2^2+\&c)+(c_0c_1+c_1c_2+\&c)(x+x^{-1})$$
$$+(c_0c_2+c_1c_3+\&c)(x^2+x^{-2})+\&c\}$$

$$=a^{-2m}\{(c_0^2+c_1^2+c_2^2+\&c)+2(c_0c_1+c_1c_2+\&c)\cos a$$
$$+2(c_0c_2+c_1c_3+\&c)\cos 2a+\&c\}.$$

COR. 1. Hence, as in [50 Cor.], we may expand $(1-p\cos a)^{-m}$.

COR. 2. If $m=1$, the result of [51] becomes

$$\begin{aligned}(a^2-2ab\cos a+b^2)^{-1} &= a^{-2}(1-nx)^{-1}(1-nx^{-1})^{-1}\\ &= a^{-2}(1+nx+n^2x^2+\&c)\,(1+nx^{-1}+n^2x^{-2}+\&c)\\ &= a^{-2}\{(1+n^2+n^4+\&c)+(n^1+n^3+n^5+\&c)\,2\cos a\\ &\qquad +(n^2+n^4+n^6+\&c)\,2\cos 2a+\&c\},\\ &= a^{-2}\{\frac{1}{1-n^2}(1+2n\cos a+2n^2\cos 2a+\&c)\}:\end{aligned}$$

hence $\dfrac{1}{a^2-2ab\cos a+b^2}=\dfrac{1}{a^2-b^2}\{1+2\dfrac{b}{a}\cos a+2\dfrac{b^2}{a^2}\cos 2a+\&c\}$,

from which, as before, we may get the series for $\dfrac{1}{1-p\cos a}$.

COR. 3. Since, in any triangle, $c^2=a^2-2ab\cos C+b^2$, we may thus get expressions for c^{-2} and, generally, for c^{-2m} in terms of a, b, C.

52. We have seen that, if r or $+r$ represent a line drawn in any one given direction, then $-r$ will represent a line of the same length, drawn in the opposite direction. This, however, is only a particular case of a far more comprehensive rule of Symbolical Algebra, namely, that, if $+r$ represent a line drawn in any given direction, then a line of the same length, inclined to the former at an angle θ, may be represented by $(\cos\theta+\sqrt{-1}\sin\theta)\,r$.

The truth of the above principle may be best inferred, as in (7), from the correctness of complicated results, obtained by means of it. We shall here, however, illustrate it by some Examples of its application.

And first we may shew that the above formulæ includes the cases already ascertained. For, if $\theta=\pi$, that is, if the second line be drawn in the *opposite* direction to the primary, (to that, namely, expressed by $+r$,) then the formula becomes $(\cos\pi+\sqrt{-1}\sin\pi)\,r$ or $-r$; and, if $\theta=0$ or 2π, that is, if the second line be drawn in the *same* direction with the primary, then the formula becomes $(\cos 0+\sqrt{-1}\sin 0)\,r$ or $(\cos 2\pi+\sqrt{-1}\sin 2\pi)\,r$, that is, it becomes, in each case, $+r$, as it should be.

If $\theta=\frac{1}{2}\pi$, we get $(\cos\frac{1}{2}\pi+\sqrt{-1}\sin\frac{1}{2}\pi)\,r$ or $+r\sqrt{-1}$, as the Symbolical expression for a line, of length r, in the position AD, at right angles to AB; and so, if $\theta=\frac{3}{2}\pi$, (or $-\frac{1}{2}\pi$,) we get $-r\sqrt{-1}$, as the expression for a line, of length r, in the position AD', *directly opposite* to the former.

Once more, supposing $(\cos\theta + \sqrt{-1}\sin\theta)\, r$ to represent a line, of length r, inclined to the primary at an angle θ, then a line, of length r, inclined to the former at an angle ϕ, would, by the same principle, be expressed by

$$(\cos\phi + \sqrt{-1}\sin\phi)\,(\cos\theta + \sqrt{-1}\sin\theta)\, r = \{\cos(\phi+\theta) + \sqrt{-1}\sin(\phi+\theta)\},$$

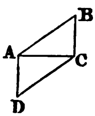
by Demoivre's Theorem—which should be the case, since this last line is inclined to the primary at an angle $(\phi + \theta)$.

Lastly, if the circumference of a circle, beginning from any point B, be divided into n equal parts in P_1, P_2, &c, P_n (or B), then, if the radius $AB = r$, we shall have AP_1, AP_2, &c, expressed (by this principle) by

$$\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}, \quad \cos\frac{4\pi}{n} + \sqrt{-1}\sin\frac{4\pi}{n}, \text{ \&c.}$$

or [23 Cor.] by ar, a^2r, &c, till we come round again to AP_n (or AB) $= a^n r$, or $+r$, as it should be.

63. With this extension of the mode of expressing symbolically both the magnitude and direction of straight lines, we shall find that the line joining any two points is the symbolical sum of any number of lines that can be drawn (forming the sides of a polygonal figure) from one point to the other.



Ex. 1. In the triangle ABC, right-angled at C, denoting the angle BAC by θ, and the *length* of AB by r, then AB will be expressed symbolically, with reference to AC as the primary or initial line, by

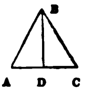
$$(\cos\theta + \sqrt{-1}\sin\theta)\, r = r\cos\theta + \sqrt{-1}\, r\sin\theta = AC + CB.$$

Or, if we denote the *length* of AC by a, and that of BC by b, then the symbolical expression for AB, namely $(\cos\theta + \sqrt{-1}\sin\theta)\, r$ becomes

$$\left\{\frac{a}{\sqrt{(a^2+b^2)}} + \sqrt{-1}\,\frac{b}{\sqrt{(a^2+b^2)}}\right\}\sqrt{(a^2+b^2)} = a + b\sqrt{-1} = \text{(as before) } AC + CB.$$

It appears then that the symbolical expression $a + b\sqrt{-1}$ represents a line, whose length is $\sqrt{(a^2 + b^2)}$, inclined to the primary line at an angle $\theta = \tan^{-1}\frac{b}{a}$.

Ex. 2. Completing the parallelogram $ABCD$ in Ex. 1, we have $AD = -\sqrt{-1}\, b$, and also DC (which is drawn from D, parallel to AB and in the *same* direction with it,) $= AB = a + b\sqrt{-1}$; and, *consequently*, $AD + DC = a = AC$.



Ex. 3. More generally, let ABC be a triangle, and draw BD perpendicular on AC; then by (Ex. 2) we have $AB + BC = (AD + DB) + (BD + DC) = AD + DC = AC$, the symbolical sum of the lines DB and BD being zero.

Ex. 4. Hence $AB + BC + CA = 0$, or the symbolical sum of the three sides of a triangle, taken in order, is *zero*:—a result which we may prove *directly*, as follows, using the figure in Ex. 3.

Observing that AB makes with the initial-line the angle A, BC the angle $-C$, (as we find by prolonging BC and AC,) and CA the angle π, we have $AB + BC + CA$

$$= \{\cos A + \sqrt{-1} \sin A\} c + \{\cos(-C) + \sqrt{-1} \sin(-C)\} a + \{\cos\pi + \sqrt{-1} \sin\pi\} b$$
$$= (c \cos A + a \cos C - b) + \sqrt{-1} (c \sin A - a \sin C) = 0,$$

by (100 Ex. 1) and (105).

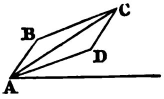
Ex. 5. Since each of the sides in the preceding Example may be broken up into two, and these again, each into two, &c, it is plain that the symbolical sum of the sides of *any* polygon, taken in order, will be zero: or, if the lengths of the sides be a, b, c, &c, making angles α, β, γ, &c, with the primary line, then

$$(\cos\alpha + \sqrt{-1} \sin\alpha) a + (\cos\beta + \sqrt{-1} \sin\beta) b + (\cos\gamma + \sqrt{-1} \sin\gamma) c = \&c = 0,$$

whence we get

$$a \cos\alpha + b \cos\beta + \&c = 0, \text{ and } a \sin\alpha + b \sin\beta + \&c = 0.$$

Ex. 6. In the annexed figure, let the lengths of AB, AD, be r and r', and let the angles made by them with the initial-line be θ, θ', respectively.



Then $AB = DC = (\cos\theta + \sqrt{-1} \sin\theta) r$,

$AD = BC = (\cos\theta' + \sqrt{-1} \sin\theta') r'$;

and $AC =$ (Ex. 3) $AB + BC = (r \cos\theta + r' \cos\theta') + \sqrt{-1} (r \sin\theta + r' \sin\theta')$, which (Ex. 1) represents a line, whose length is

$$\sqrt{\{(r \cos\theta + r' \cos\theta')^2 + (r \sin\theta + r' \sin\theta')^2\}} = \sqrt{\{r^2 + r'^2 + 2rr' \cos(\theta - \theta')\}},$$

inclined to the initial-line at an angle $\phi = \tan^{-1} \dfrac{r \sin\theta + r' \sin\theta'}{r \cos\theta + r' \cos\theta'}$.

For further remarks on this subject the Student is referred to Peacock's Algebra, vol. II. chap. 31.

MISCELLANEOUS EXAMPLES.

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1. If $\sin x = \sin a \sin(x+y)$, shew that $\tan x = \dfrac{\sin a \sin y}{1 - \sin a \cos y}$.
2. Find the value of $a \cos 2\theta + b \sin 2\theta$, when $\tan\theta = \dfrac{b}{a}$.
3. Shew that $\text{chd}\,\frac{3}{2}A\ \text{chd}\,\frac{1}{2}A = \text{chd}^2 A - \text{chd}^2 \frac{1}{2}A$.
4. If $m \operatorname{vers}\theta = n \pm p \operatorname{vers}(\pi - \theta)$, find the value of $\operatorname{vers}\theta$.
5. Shew that the ratios of an English to a Foreign minute and second are $\dfrac{2.5^2}{3.3^2}$ and $\dfrac{2.5^3}{3.3^3}$ respectively.
6. Given the area of a plane triangle in terms of its sides to be
$$\tfrac{1}{4}\sqrt{(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)},$$
shew what this expression becomes, when C is a right angle.
7. Shew that the areas of all triangles, described about the same circle, are proportional to their perimeters.
8. The Moon's distance from the Earth being sixty times the Earth's radius, shew that the Earth's radius subtends at the Moon an angle $0^\circ.95$.
9. The sides of a triangle are 13, 12, and 5 yards: find its area.
10. Given $\log 4.2366 = .6270227$, find the value of $(\frac{1}{8})^{\frac{187}{14}}$.
11. Shew that in any circle the chord of an arc of 108° is equal to the sum of the chords of 36° and 60°.
12. If $\cos A = \cos B \cos C$, shew that
$$\tan\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B) = \tan^2\tfrac{1}{2}C.$$
13. Express $\cos m\theta \cos n\theta \cos r\theta$ by the sum of a series of cosines of multiples of θ.
14. Shew that, in any triangle,
$$\tan A = \frac{a \sin B}{c - a \cos B} \text{ and } \cot A = \frac{b}{a}\operatorname{cosec} C - \cot C.$$
15. Prove the formula $\cot\dfrac{A}{2} = \sqrt{\dfrac{\sec A + 1}{\sec A - 1}}$.
16. Shew that
$$\cos(36^\circ + A)\cos(36^\circ - A) + \cos(54^\circ + A)\cos(54^\circ - A) = \cos 2A.$$
17. *Determine* $\sin A$ from the equation $\tan^2 A + 4\sin^2 A = 1$.

18. A river AC, whose breadth is 200 feet, runs at the foot of a tower BC, which subtends at the edge of the bank an angle BAC of 25° 10′. Find the height of the tower, having given $\text{L}\tan 64° \, 50' = 10.3280372$, $\log 5 = .6989700$, and $\log 9.397 = .9729928$.

19. Obtain $\sin A$ in terms of $\sin 2A$, when A lies between 180° and 225°.

20. Prove that $1 + \cos(2A - 2B)\cos 2B = \cos^2 A + \cos^2(A - 2B)$.

21. Shew that, if $\sin(A + B)\cos C = 2\cos(B - C)\sin A$, then $\cot A - \cot B = 2\tan C$.

22. If A and B are the two acute angles of a right-angled triangle, shew that $\sin(A - B) = -\cos 2A$, and $\sin 2A + \sin 2B = 2\cos(A - B)$.

23. If α, β, γ, denote the distances from the angular points of a triangle to the points of contact of the inscribed circle, shew that its radius $= \left(\dfrac{\alpha\beta\gamma}{\alpha + \beta + \gamma}\right)^{\frac{1}{2}}$.

24. Prove that the circular measure of 42° is .73303....

25. Shew that $\sin A \cos(B - C) - \sin B \cos(A - C) = \sin(A - B)\cos C$.

26. Find the sine, secant, and tangent of 285°.

27. Shew that $\tan^{-1}\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} + \tan^{-1}\sqrt{\tfrac{3}{2}} = \tfrac{3}{4}\pi$.

28. Prove the formula $(1 + \tan\frac{1}{2}A)^2 = 2\left(\dfrac{1 + \sin A}{1 + \cos A}\right)$.

29. Shew that $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80} = \log 2$.

30. If A, B, C, are in A. P., shew that

$$\sin A - \sin C : \cos C - \cos A :: \cos B : \sin B.$$

31. If $\tan(\theta + 45°) + \tan(\theta - 45°) = 2\tan 60°$, determine θ.

32. A man, distant 70 feet from the foot of a tower, observes it to subtend an angle of 60°, his eye being on a level with the ground: find the height of the tower, given $\log 3 = .4771213$, $\log 7 = .8450980$, $\log 1.21 = .0836586$.

33. If $\dfrac{\cos A - \sin B \sin C}{\cos B \cos C} = \dfrac{\cos D - \sin E \sin F}{\cos E \cos F}$, shew that

$$\frac{\cos B \cos C}{\cos E \cos F} = \frac{\text{vers}\, A - \text{vers}(B - C)}{\text{vers}\, D - \text{vers}(E - F)}.$$

34. Shew that, in any triangle, $\cos A + \cos B = \dfrac{2(a + b)}{c}\sin^2\frac{1}{2}C$.

35. Taking the Earth's circumference at 25,000 miles, find its diameter.

36. Shew that $\sec(\frac{1}{4}\pi+\theta)\sec(\frac{1}{4}\pi-\theta)=2\sec 2\theta$.

37. One angle of a triangle is 15°, and another is $2\frac{2}{3}$ of the third; shew that the sides opposite to these latter are as $\sqrt{3}:\sqrt{2}$.

38. The perimeter of an equilateral triangle being 27 yards, find the area.

39. *CD* is the perpendicular from the right angle *C* of a triangle upon the hypothenuse *AB*, *AC* is 108 yds, *CB* is 144 yds; find *CD*, *BD*, and *DA*.

40. Given 75° the sum of two angles and 75^g their difference, express the angles in English and Foreign measure.

41. If the right angle were divided into 100° instead of 90°, what would be the circular measure of 36°?

42. Given $\cos A=.6$, find the values of $\sin A$, $\cot A$, and $\operatorname{chord} A$.

43. Prove that $\sec 2A-\tan 2A=\dfrac{\cos A-\sin A}{\cos A+\sin A}$.

44. If α, β, γ, be any angles, shew that
$$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)=4\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha+\gamma)\sin\tfrac{1}{2}(\beta+\gamma).$$

45. Find the area of a triangle whose sides are $13\frac{1}{2}$, 18, $22\frac{1}{2}$ yds, respectively.

46. Shew that $\sin 30°\,1'=\cos 1'-\sin 29°\,59'$.

47. Find the angle whose circular measure is $\frac{5}{6}$.

48. Prove the formula $\dfrac{1+\sin A}{1-\sin A}=\tan(\frac{1}{4}\pi+\frac{1}{2}A)\cot(\frac{1}{4}\pi-\frac{1}{2}A)$.

49. Find the area of a circular sector, where the radius is 9 feet, and the sine $4\frac{1}{2}$ feet.

50. Obtain the acreage of a triangular piece of ground, the sides of which are 380, 420, and 400 yards.

51. Prove the formula $\tan\dfrac{A}{2}=\sqrt{\dfrac{2\sin A-\sin 2A}{2\sin A+\sin 2A}}$.

52. Shew that an inch will subtend an angle of 1″ nearly at a distance of 3 miles.

53. Shew that $\cot^{-1}3+\operatorname{cosec}^{-1}\sqrt{5}=\frac{1}{4}\pi$.

54. If *AD*, *BE*, *CF*, be drawn perpendicular from the angles upon the sides of a triangle, shew that the products of the alternate segments of the sides will be equal.

55. If α, β, γ, be any angles, shew that
$$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)=4\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha+\gamma)\cos\tfrac{1}{2}(\beta+\gamma).$$

56. An observer, standing on the bank of a straight canal, observes the angle (a), which the distance of a tree on the other side makes with the canal, and then walks along the bank a distance (a), until the distance of the tree makes the same angle with the canal in the other direction. Find the shortest distance of the tree from the bank.

57. Find the area of a trapezoid, of which the parallel sides are 750 and 1225 links, and the distance between them 1540 links.

58. Prove the formula $\tan(\frac{1}{4}\pi - \frac{1}{2}\theta) + \cot(\frac{1}{4}\pi - \frac{1}{2}\theta) = 2\sec\theta$.

59. If $\sin\alpha = m\sin\beta$, and $\tan\alpha = n\tan\beta$, shew that $\cos\alpha = \sqrt{\dfrac{m^2-1}{n^2-1}}$.

60. Shew that in any triangle $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = \dfrac{a^2+b^2}{a^2+c^2}$.

61. If the right angle were divided into *sixty* degrees instead of *ninety*, what would be the circular measure of 75° ?

62. If s, s', be the secants of two arcs, t, t', their tangents, then

$$s^2s'^2 - s^2 : s^2s'^2 - s'^2 :: t^2t'^2 + t'^2 : t^2t'^2 + t^2.$$

63. Given $\tan\theta = \dfrac{2\sqrt{(ab)}}{a-b}$; find the values of $\sin\theta$ and $\operatorname{vers}\theta$.

64. From $\log 2 = .3010300$, and $\log 3 = .4771213$, find the logs of 72, 45, 1800, .00024.

65. Shew that $\cot\theta - \tan\theta = 2\sqrt{\dfrac{1+\cos 4\theta}{1-\cos 4\theta}}$.

66. Prove that $\sin 48° = \frac{1}{8}\{\sqrt{(10+2\sqrt{5})} + \sqrt{3}(\sqrt{5}-1)\}$.

67. Shew that the area of a triangle

$$= \tfrac{1}{2}(a^2+b^2+c^2)\frac{c\sin A\sin B}{a\sin A + b\sin B + c\sin C}.$$

68. From two points, A, B, in the diameter produced of a semicircle, tangents AP, BQ, are drawn to the circumference: find the radius CP, given $\angle CAP = \alpha$, $\angle CBQ = \beta$, and $AB = a$.

69. Shew that $\dfrac{1}{a+b\cos x} = \dfrac{\sec^2\frac{1}{2}x}{(a+b)+(a-b)\tan^2\frac{1}{2}x}$.

70. If a be a side of a regular polygon of n sides, and R, r, the radii of its circumscribed and inscribed circles, shew that

$$R + r = \tfrac{1}{2}a\cot\frac{\pi}{2n}.$$

71. Shew that the line (CD) drawn from the angle C of a triangle to the base, and bisecting the angle, $= \dfrac{2ab}{a+b}\cos\frac{1}{2}C$.

72. Find the number whose log is -1.8753145, having given
$$\log 1.3325 = .1246672,\quad \log 1.3326 = .1246998.$$

73. Prove that $\sec\theta = \dfrac{2}{\sqrt{\{2+\sqrt{(2+2\cos 4\theta)}\}}}$.

74. If p, q, r, be the perpendiculars upon the sides a, b, c, of a triangle from the opposite angles, shew that
$$\frac{p^2}{qr}+\frac{q^2}{pr}+\frac{r^2}{pq}=\frac{bc}{a^2}+\frac{ac}{b^2}+\frac{ab}{c^2}.$$

75. Shew that $\text{vers}\,(2\pi-\theta) - \text{covers}\,(\tfrac{3}{2}\pi-\theta) - \text{suvers}\,(\pi-\theta)$
$$+\ \text{suvers}\,(\theta-2\pi) + \text{covers}\,(\theta-\tfrac{3}{2}\pi) - \text{suvers}\,(\theta-\pi) = 0.$$

76. Shew that the area of a triangle
$$= \tfrac{1}{4}(a+b+c)^2 \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C.$$

77. Find in degrees the angle whose circular measure is .7854.

78. The number of sides of one regular polygon exceeds that of another by 1, and an angle of the first exceeds that of the second by 4°: find the number of sides in each.

79. Given $\tan^3\theta = \dfrac{b}{a}$: find the value of $a\sin\theta + b\cos\theta$, and of $a\sec\theta + b\,\text{cosec}\,\theta$.

80. Obtain the circular measure of $1^g\, 3^`\, 75^{``}$, and express the angle in degrees, &c.

81. An arc of 120° in one circle is equal to the whole circumference of another: compare their radii, and find how many degrees of the circumference of the smaller circle are equal in length to the radius of the greater.

82. How many acres are there in a triangular field, whose sides are 13, 14, and 15 chains?

83. Shew that $\text{vers}\,(\dfrac{m}{m+n}\pi-\theta) + \text{vers}\,(\dfrac{n}{m+n}\pi+\theta) = \text{vers}\,\pi$.

84. Prove that $\sin 41^\circ + \sin 67^\circ - \sin 31^\circ - \sin 77^\circ = \sin 5^\circ$.

85. The diameter of a circle is divided in the ratio of $1\tfrac{2}{3} : 2\tfrac{1}{2}$, and circles are described on the segments as diameters: compare the areas of the three circles.

86. The sum of two angles is m degrees, and the difference n grades: express the greater in degrees, and the less in grades.

87. If, from the angles of a triangle ABC, three lines be drawn to meet in a point O, shew that
$$\sin ABO \,.\, \sin BCO \,.\, \sin CAO = \sin CBO \,.\, \sin ACO \,.\, \sin BAO.$$

88. At 225 feet from the foot of a steeple, the elevation was exactly half what it was at 100 feet from it: find the height of the steeple.

89. If $\tan A + \sin A = m$ and $\tan A - \sin A = n$, determine $\cos A$; and shew, by eliminating A, that $(m^2 - n^2)^2 = 16mn$.

90. Solve the equation

$$\sin(x+a) + \cos(x+a) = \sin(x-a) + \cos(x-a).$$

91. Prove that $1 - \dfrac{\sin^2 A}{1+\cot A} - \dfrac{\cos^2 A}{1+\tan A} = \frac{1}{2}\sin 2A$.

92. If a foot-square of card-board be divided into four equal squares, by lines bisecting the sides at right angles, how much of it will be left, when the four circles, that can be inscribed in these four squares, are cut out?

93. Find the area of a triangular field, whose sides are 70, 80, 90, yards, respectively.

94. The sides of a triangle being as 3, 4, 5, apply the expression for the tangent of half an angle in terms of the sides to find the largest angle.

95. If C, C', be the two values of the third angle in the ambig. case, when a, b, A, are given, and $b > a$, then $\tan A = \cot\frac{1}{2}(C+C')$.

96. Find the length of the arc, which subtends an angle 6° 12′ 36″ in a circle whose radius is 100 yards.

97. A mill-sail is 7 yds in length, and goes round uniformly 10 times in a minute. At what rate per hour does its extremity move?

98. Prove the formula $\tan(A+30°)\tan(A-30°) = \dfrac{1-2\cos 2A}{1+2\cos 2A}$.

99. Find the number of degrees in the arcs AE, $BCDF$, in the figure of Euc. IV. 16: and, if BC be joined, find the angles of the triangle ABC.

100. How many cubic inches of iron will be required to form a garden roller, which is half-an-inch thick, with an outer circumference of $5\frac{1}{2}$ feet, and a width of $3\frac{1}{2}$ feet? $(\pi = \frac{22}{7})$.

101. Shew that

$$2\cos A = \sqrt{\{2 + \sqrt{(2 + \sqrt{(2 + \ldots\ldots + \sqrt{(2 + 2\cos 2^n A)}}}\},$$
$$2\sin A = \sqrt{\{2 - \sqrt{(2 + \sqrt{(2 + \ldots\ldots + \sqrt{(2 + 2\cos 2^n A)}}}\}.$$

102. Eliminate ϕ between the two equations

$$x = 3\cos\phi + \cos 3\phi, \quad y = 3\sin\phi - \sin 3\phi.$$

103. Find the value of z in the equation $2\sqrt{(z^2 - z^4)} = \sin A$.

104. If $\tan\theta = \dfrac{b\sin\phi}{a + b\cos\phi}$, shew that $\tan(\phi - \theta) = \dfrac{a\sin\phi}{b + a\cos\phi}$.

105. In any triangle, shew that $a^2 \sin 2B + b^2 \sin 2A = 2ab \sin C$.

106. Prove that $\sin\alpha \cos(\beta-\gamma) - \sin\beta \cos(\alpha-\gamma) = \sin(\alpha-\beta) \cos\gamma$.

107. If r be the radius of the inscribed circle of a triangle, shew that the product of the three perpendiculars from the angles on the sides will be $\dfrac{(a+b+c)^3}{abc} r^3$.

108. Given Log-sin 59° 37′ 40″ = 9.9358894, and Diff. for 10″ = 124, find A' from the equation Log-sin A' = 9.9358921, and find also Log-sin A'', where A'' = 59° 37′ 45″.34.

109. Prove that $\sin^{-1}\sqrt{\dfrac{x}{a+x}} = \tan^{-1}\sqrt{\dfrac{x}{a}} = \frac{1}{2}\cos^{-1}\dfrac{a-x}{a+x}$.

110. Find the sine, cosine, and tangent of $(m+\frac{1}{2})\pi + \theta$.

111. Find the value of θ when $\sin(\theta+\frac{1}{4}\pi) \sin(\theta-\frac{1}{4}\pi) = \sin^2 \frac{1}{6}\pi$.

112. Given $\log 1\frac{1}{4} = .0969100$ and $\log .\dot{1} = \bar{1}.0457575$, find the logs of $2\frac{1}{2}$, $2\frac{1}{4}$, and $.\dot{2}$.

113. AB, AC, both measured in the same direction from A, are the sides of a regular octagon and heptagon, inscribed in the same circle: find the number of degrees in the angle BAC, and find also the interior angle of a regular figure, one of whose sides shall be BC.

114. Prove that $\tan(\alpha+\beta) = \dfrac{\sin^2\alpha - \sin^2\beta}{\sin\alpha \cos\alpha - \sin\beta \cos\beta}$.

115. From the arc AB of a circle, whose centre is O, AC is cut off equal to the sine of AB: shew that the sector BOC is equal to the segment ACB.

116. If p be the perpendicular from the angle C of a triangle upon the opposite side, shew that

$$\frac{a \sin A + b \sin B + c \sin C}{bc \cos A + ac \cos B + ab \cos C} = \frac{2p}{ab}.$$

117. If R, r, be the radii of the circles described about and in the triangle ABC, shew that the area of the triangle

$$= Rr(\sin A + \sin B + \sin C).$$

118. If $\cos(\alpha+\beta) \sin(\gamma+\delta) = \cos(\alpha-\beta) \sin(\gamma-\delta)$, shew that

$$\tan\delta = \tan\alpha \tan\beta \tan\gamma.$$

119. In any triangle, shew that $\dfrac{\sin A}{\sin B} = \dfrac{\cos A \cos C + \cos B}{\cos B \cos C + \cos A}$.

120. Apply the last result to shew that $\sin A : \sin B :: a : b$, assuming the expression for the cosine of an angle of a triangle in terms of the sides.

121. Shew that, in any plane triangle, $c = (a - b)\sec\phi$, where ϕ is given by the formula $\tan\phi = \dfrac{2\sqrt{(ab)}}{a-b}\sin\frac{1}{2}A$. What objection would there be to using this formula, if $a = b$ nearly?

122. Find the value of A in the equation

$$\tan(A + 45°) + \tan(A - 45°) = 2\tan 60°.$$

123. Prove (by means of the Binomial Theorem) that

$$(1 + 2\cos^2\theta + 3\cos^4\theta + \&c)(1 - 2\tan^2\theta + 3\tan^4\theta - \&c) = \cot^4\theta.$$

124. Given $\cos nA + \cos(n-2)A = \cos A$: find A in general terms.

125. Given that $\sin(\beta + \theta) + \sin a(\sin^2\beta - \sin^2 a) + \sin\{2a - (\beta - \theta)\}$ $= (\cos^2\beta - \cos^2 a)\sin(a + 2\theta)$, shew that $\sec\theta = \sin(a + \beta)\tan(a - \beta)$.

126. Given $\log 2 = .3010300$, and $\log 3 = .4771213$, find the logs of $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{8}}$, and $(3\frac{1}{5})^{-\frac{1}{3}}$.

127. If $a + \beta + \gamma = \pi$, shew that

$$\tan\left(\frac{2^n-1}{2^n}\cdot\frac{\pi}{3} + \frac{a}{2^n}\right) + \tan\left(\frac{2^n-1}{2^n}\cdot\frac{\pi}{3} + \frac{\beta}{2^n}\right) + \tan\left(\frac{2^n-1}{2^n}\cdot\frac{\pi}{3} + \frac{\gamma}{2^n}\right)$$

= the product of the same three tangents.

128. Three objects A, B, C, form a triangle, whose angles are as the numbers 1, 4, 1. B is inaccessible from A or C; but an observer, walking from A towards C, measures a base AD (a feet) and observes the angle BDC; he then advances b feet farther to E, and observes the angle BEC to be the supplement of the former angle. Find the distance of B from A: and shew that $AB = DE\tan BDE$.

129. In any plane triangle, $c = a\cos B \pm \sqrt{(b^2 - a^2\sin^2 B)}$.

130. Given $\log 2 = .30103$, find the logs of 5, .016, $\frac{1}{2}$, 6.25, $1\frac{3}{5}$, $15\frac{5}{8}$.

131. Given $\cot^{-1}(x - 1) - \cot^{-1}(x + 1) = \frac{1}{12}\pi$, determine x.

132. Prove that $\dfrac{\tan A + \sec A}{\cot A + \operatorname{cosec} A} = \tan(45° + \frac{1}{2}A)\tan\frac{1}{2}A$.

133. If $\dfrac{m\tan(a - x)}{\cos^2 x} = \dfrac{n\tan x}{\cos^2(a - x)}$, then $x = \frac{1}{2}\{a - \tan^{-1}(\dfrac{n-m}{n+m}\tan a)\}$.

134. The area of a regular polygon, inscribed in a circle: that of the corresponding circumscribed polygon :: 3 : 4; find the number of sides.

135. Eliminate θ between the equations

$$(a+b)\tan(\theta - \phi) = (a - b)\tan(\theta + \phi) \text{ and } a\cos 2\phi + b\cos 2\theta = c.$$

136. One angle of a triangle is 18° and another 54°, and the least side is 96 feet: find the greatest side.

7. If $\tan\theta = \sec a - \tan a$, find θ in terms of a.

8. The excess of the sine above the versed-sine, in angles less than 90°, is greatest when the angle is 45°. Prove this, and find the *maximum* value of the excess.

9. Find the changes in sign and magnitude of the quantity $\sin\theta + \cos\theta$, while θ changes from 0° to 360°.

0. Find, in a form adapted to logarithmic computation, the area of a trapezium, whose two *parallel* sides are a and b, and c the distance between them.

1. Determine $\cos\phi$ from the equation $\tan\frac{1}{2}\phi = \text{cosec}\,\phi - \sin\phi$.

2. The angles of an inscribed quadrilateral, taken in order, when multiplied by 1, 2, 2, 4, respectively, are in A. P.: find them.

3. Prove that $1 \pm \sin\theta = 2\sin^2(\frac{1}{4}\pi \pm \frac{1}{2}\theta)$.

4. Determine A from the equation $1 + 2\sin 4A = 4\sin 3A \cos A$.

5. Shew that the equation $\sin\theta \sin(2a + \theta) + n\cos^2 a = 0$ cannot hold, unless $\sec a > \sqrt{(1 + n)}$.

6. If chord $\theta = \cos\theta$, shew that $\theta = 2\cos^{-1}\sqrt{\cos\frac{1}{6}\pi}$.

7. A person, from the top of a tower, observes the angles of depression (α, β) of two objects, whose distance (a) from each other he knows, and which lie just in a line with the tower. Determine its height.

8. Find a from the equation $\tan\frac{1}{2}a = \text{cosec}\,a$.

9. Shew that

$$\{\tan(A+45°)+\tan(A-45°)\}\div\{\cot(45°+A)+\cot(45°-A)\}=\sin 2A.$$

0. Eliminate θ between the equations

$$m = \text{cosec}\,\theta - \sin\theta, \quad n = \sec\theta - \cos\theta.$$

1. If $(x - a)\cos\delta + c\sin(l - \delta) = 0$, $(y - b)\cos\delta + c\cos(l - \delta) = 0$, prove that $(x - a)\sin l + (y - b)\cos l + c = 0$.

2. Given $\beta = 4a^2$, where a is the circular measure of 3′ 47″, shew that the value of β is 1″, very nearly.

3. Solve the equations (i) $\cos A = \cos 2A$, (ii) $\tan 2A = 3\tan A$.

4. If a triangle ABC be formed, by joining the centres of three circles which touch one another, and if s be the semi-sum of its sides a, b, c, then the radii of the circles, whose centres are A, B, C, respectively, will be $s - a$, $s - b$, $s - c$.

5. At the distance of 50 miles from a tower, its top just appeared in the horizon; determine its height, having given the Earth's *diameter* to be 7964 miles.

156. Shew that $1+\sin^2 A\cot^2 A-2\sin A\cot A+\cos^2 A\tan^2 A=\text{chd}^2 A$.

157. Given $1-\sin a=2\sin^2(x-\frac{1}{2}a)$, find x.

158. If the side of a regular inscribed polygon be an m^{th} part of the radius, shew that the secant of the angle subtended by it at the centre will be $\dfrac{2m^2}{2m^2-1}$.

159. Given tab-sin 18° 1′ = 3092.936, find tab-sin 18° 0′ 23″.

160. Shew that $\cos^2\theta+\dfrac{\sin\theta(1+\sin\theta)(1-\tan\frac{1}{2}\theta)}{1+\cot\frac{1}{2}\theta}=\cos\theta$.

161. Find the area of a circle, traced by a pair of compasses, the length of whose legs is 6 inches, and the angle between them 120°.

162. Solve the equation $\tan A(\tan 2A+\cot A)=2$.

163. Prove that the square of the side of an inscribed pentagon equals the sum of the squares of the sides of a hexagon and decagon, inscribed in the same circle.

164. A general, having got possession of a town, ascends a tower, and sees a river, whose breadth he wishes to ascertain. He first determines the height of the tower above the horizontal plane to be 210 feet, and then observes the angles of depression of the two banks of the river, where it crosses the plain directly before him, to be 23° and 25°. Shew how he may find the width of the stream, having given

$$L\sin 2^\circ=8.5428192,\quad L\sin 23^\circ=9.5918780,\quad L\sin 25^\circ=9.6259483,$$
$$\log 3=.4771213,\quad \log 7=.8450980,\quad \log 4.43=.6472122.$$

165. Determine ϕ when $\tan\phi+2\cot 2\phi=\sin\phi(1+\tan\phi\tan\frac{1}{2}\phi)$.

166. Prove that $\cos^6 a-\sin^6 a=\cos 2a(1-\frac{1}{4}\sin^2 2a)$.

167. In any triangle, shew that

$$a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C,$$

where R is the radius of the circumscribing circle.

168. The sides a, b, c, of a triangle are as the numbers 4, 5, 6: find the angle B, having given $\log 2=.3010300$, $L\cos 27^\circ 53'=9.9464040$, $L\cos 27^\circ 54'=9.9463371$.

169. Obtain the numerical value of sec 67° 30′.

170. Shew that $\dfrac{\sin 2a+\cos 2a}{(\cos a-\sin a)-(\cos 3a-\sin 3a)}=\frac{1}{2}\operatorname{cosec} a$.

171. Put the expression $a\cos\theta+b\cos(\theta+a)$ into the form $k\cos(\theta+\beta)$, determining the values of k and β.

172. Two observers, a mile apart from each other, on the same side of a balloon, and in the same vertical plane with it, find its angles of elevation at the same moment to be 30° and 75°. Find the height of the balloon.

173. Shew that

$$\cot\theta + \cot 2\theta + \cot 4\theta = \operatorname{cosec} 4\theta(2 + 2\cos 2\theta + 3\cos 4\theta).$$

174. Prove that the following rule for finding the distance of the horizon at sea is very nearly correct: take the number of feet in the height of the station above the level of the sea, and increase it by half that number; then the square root of this quantity will give the distance of the horizon in miles.

175. The sides of a triangle are to each other as 9 : 7, and the included angle is 64° 12′: determine the other angles, having given $\log 2 = .3010300$, $L \tan 57° 54' = 10.2025255$, $L \tan 11° 16' = 9.2993216$, $L \tan 11° 17' = 9.2999804$.

176. Express $\sin^3\theta \cos^2\theta$ by a series of sines of multiples of θ.

177. Find the distance from the eye, at which a circular disc of six inches diameter must be placed, so as just to cover the Moon, whose apparent diameter is 0°.95.

178. Transform the equation $\sqrt{(2 \sec A)} = \frac{1}{5}\sqrt[3]{\frac{\cos^2 B}{\sin C}}$ into the equation between the tabular logarithms of the quantities.

179. Prove that

$$\operatorname{cosec}^2\theta + \operatorname{cosec}^2\phi - 2\operatorname{cosec}\theta \operatorname{cosec}\phi \cos(\theta - \phi) = (\cot\theta \sim \cot\phi)^2.$$

180. In making the circuit of a field, I walked 30 yards due S, then 60 yards due E, then 110 yards due N, and then straight to the place of first starting. Find the circumference and area of the field.

181. If lines be drawn from the angles of a triangle ABC to the centre of the inscribed circle, cutting its circumference in D, E, F, shew that the angles of the triangle DEF are, respectively, $\frac{1}{4}(\pi + A)$, $\frac{1}{4}(\pi + B)$, $\frac{1}{4}(\pi + C)$.

182. Eliminate θ between the equations

$$\cos(\phi - \theta + a)\cos(\theta - a) = \cos(\phi - \theta - a)\cos(\theta + a) = c.$$

183. Write down the values of $\sec \frac{1}{3}m\pi$, m being an integer.

184. Express $\tan A$ in terms of $\tan 2A$, and exhibit by a geometrical figure the two values of $\tan A$.

185. Solve *the equation* $8^x . 125^{2-x} = 2^{4x+3} . 5^x$, given $\log 2 = .30103$.

186. The hypothenuse AB of a right-angled triangle is trisected in D, E: prove that the sum of the squares of the sides of the triangle CDE is $\frac{2}{3}c^2$.

187. If the fraction $\dfrac{a\cos(\theta+\alpha)+b\sin\theta}{a'\sin(\theta+\alpha)+b'\cos\theta}$ remain the same for all values of θ, shew that $aa'-bb'=(a'b-ab')\sin\alpha$.

188. If the tangents of the semiangles of a triangle be in A. P., shew that the cosines of the angles will be also in A. P.

189. Given $\dfrac{\sin\theta}{\theta}=\dfrac{2165}{2166}$, shew that θ contains 3° nearly.

190. Given $\text{L}\sin 17^\circ = 9.4659353$, $\text{L}\sin 17^\circ\, 1' = 9.4663483$, find $\text{L}\sin 17^\circ\, 12''$, and the angle whose L sin is 9.4659894.

191. If $f(x)=a\cos x\theta+b\sin x\theta$, shew that

$$f(x+2)-2\cos\theta\,.\,f(x+1)+f(x)=0.$$

192. If $\sin\phi=m\tan\theta$ and $\sin\theta=n\tan\phi$, find $\cos\theta$ in terms of m and n.

193. Prove that

$$\frac{\sin\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\sin\beta}{\sin(\beta-\gamma)\sin(\beta-\alpha)}+\frac{\sin\gamma}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=0.$$

194. Find the radius of a circle, when six inches is the length of the versed-sine of an arc whose chord is a yard.

195. The side BC of a triangle is bisected in D: shew that

$$\cot BAD \sim \cot CAD = \cot B \sim \cot C.$$

196. Find A in the equation $\text{L}\cos A = 9.9694579$,

given $\text{L}\cos 21^\circ\, 15' = 9.9694196$, and Diff. for $1' = 491$:

find also $\text{L}\,\text{cosec}\, 68^\circ\, 45'\, 24''$.

197. If $\frac{1}{2}(\tan\theta+\cot\theta)=\dfrac{m^2+n^2}{m^2-n^2}$, determine $\tan\theta$ and $\cos 2\theta$.

198. Prove that $\tan A$, $\tan B$, $\tan C$, are in H. P., if

$$\cos(A-C)\cos B=\cos(A-B+C).$$

199. Find the general value of A, when $\cos A+\sin A=\sqrt{2}$.

200. One angle of a triangle is 32° 15′, the sides including another are 468 and 320 yards; find the latter angle, given

$\log 2 = .3010300$, $\log 3 = .4771213$, $\log 13 = 1.1139434$,
$\text{L}\,\text{cosec}\, 32^\circ\, 15' = 10.2727724$, $\text{L}\sin 21^\circ\, 23' = 9.5621316$,
$\text{L}\sin 51^\circ\, 18' = 9.8923236$.

201. Determine θ from the equation $\cos\theta+\cos 2\theta+\cos 3\theta=0$.

202. Shew that any triangle may be divided into four others, *three of which* shall be isosceles.

3. If A', B', C', be the angles subtended by the sides of a triangle at the centre of its inscribed circle, shew that
$$4 \sin A' \sin B' \sin C' = \sin A + \sin B + \sin C.$$
4. At 120 feet distance from the foot of a steeple, the angle of elevation of the summit was found to be 60° 14″. Find its height having given $\tan 60° \, 1' = 1.7332149$.
5. Find the general value of A when $\cos 2A = \sin A$.
6. Shew that $A + \tan^{-1}(\cot 2A) = \tan^{-1}(\cot A)$.
7. If the sides of a triangle ABC be divided in D, E, F, so that $\frac{AF}{BF} = \frac{BD}{CD} = \frac{CE}{AE} = \frac{1}{n-1}$, prove that the area of the triangle formed by joining the points of section : area of original triangle $:: n^2 - 3n + 3 : n^2$.
8. At the ends of a horizontal base ($AB = a$), whose bearing (ϕ) from the North is known, the angles (α, β) are observed, subtended by the lines (BC, AC) which join the two ends of the base with the foot (C) of a certain pillar. Given also (θ) the elevation of the pillar at the end A, find its height and bearing from the North.
9. From the top of the Peak of Teneriffe, the dip of the horizon is found to be 1° 58′. If the Earth's radius be taken as 4000 miles, find the height of the mountain, given $\sec 1° \, 58' = 1.0005894$.
10. Given $a = 55$, $b = 40$, $A = 54° \, 10'$, $L \sin A = 9.9088727$, $L \sin 36° \, 7' = 9.7704332$, $L \sin 36° \, 8' = 9.7706063$: find B.
11. If l be the length of the line, drawn from the vertex C of a triangle to bisect the base ($2c$), shew that the area of the triangle $= \frac{1}{2}(l^2 - c^2) \tan C$.
12. The tangents of the angles of a triangle are as the numbers 1, 2, 3: find them, and shew that if p, q, r, be the perpendiculars upon the opposite sides from the angles A, B, C, then $pqr = \frac{3}{5}abc$.
13. Given $\cos nA + \cos(n-2)A = \cos A$, determine A.
14. Shew that $\cos^2 \varsigma + \cos^2(\varsigma - \theta) + \cos^2(\varsigma - \phi) + \cos^2(\varsigma - \psi)$
$$= 2 + 2 \cos\theta \cos\phi \cos\psi, \text{ if } 2\varsigma = \theta + \phi + \psi.$$
15. Find the tangents of 36° and $37\frac{1}{2}$°.
16. Adapt $a \cos\theta \pm b \sin\theta$ to logarithmic computation.
17. The sides a, b, c, of a triangle are as the numbers 4, 5, 6: shew that $\cos B = \cos^2 A$, and, having given $L \cos 55° \, 46' = 9.7501723$, $L \cos 55° \, 47' = 9.7499866$, find the angle B.

218. If c be the chord of a quadrantal arc, a its radius, a' the radius of the circle inscribed in the quadrant, s that $a + a' = c$.

219. An equilateral triangle and a regular hexagon have same perimeter: shew that the areas of their inscri circles are as 25 : 27.

220. Prove that the area of a triangle

$$= \tfrac{1}{4}(a^2 + b^2 - c^2) \cot \tfrac{1}{2}(A + B - C).$$

221. Given $\cot A \sin C = \cot a \sin b - \cos b \cos C$, find C in a fo adapted to logarithmic computation.

222. Given $\sin 4\theta + 4 \sin 3\theta \cos\theta = 0$; find $\tan\theta$.

223. If the hypothenuse (AB) of a right-angled triangle divided in D by a line which bisects the right angle, t

$$AD : BD :: 1 - \tan \tfrac{1}{2}(A - B) : 1 + \tan \tfrac{1}{2}(A - B).$$

224. Solve the equation $2 \cos 2A = 2 \sin A + 1$.

225. From a station B, at the base of a mountain, its summi is seen at an elevation of 60°; and, after walking one towards the summit, up a plane making an angle of with the horizon, to another station C, the angle BC observed to be 135°. Find, in feet, the height of the mo tain above the horizontal plane at B.

226. If $\sec a \sec\theta + \tan a \tan\theta = \sec\beta$,

$$\text{then } \sec a \tan\theta + \tan a \sec\theta = \tan\beta.$$

227. Shew that $\sin A + \cos A + \sin B + \cos B$

$$= 2\sqrt{2} \cos\{45° - \tfrac{1}{2}(A + B)\} \cos \tfrac{1}{2}(A - B).$$

228. Prove that $\cos^4 \frac{1}{8}\pi + \cos^4 \frac{3}{8}\pi + \cos^4 \frac{5}{8}\pi + \cos^4 \frac{7}{8}\pi = \frac{3}{2}$.

229. The elevation of a steeple, standing upon a horizontal pl is observed, and, at a station a feet nearer to it, its eleva is found to be the *complement* of the former. On advan in the same direction b feet nearer still, the elevatio found to be *double* of the first. Shew that the heigh the steeple is $\sqrt{\{(a + b)^2 - \frac{1}{4}a^2\}}$.

230. Shew that $\sqrt{\text{vers}\,\theta} = \dfrac{\sin\theta}{\sin(45° - \frac{1}{2}\theta) + \cos(45° - \frac{1}{2}\theta)}$.

231. If $A + B + C = 180°$, shew that

$$\cos \tfrac{1}{2}A + \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}C = 4 \cos \tfrac{1}{4}(A + B) \cos \tfrac{1}{4}(B + C) \cos \tfrac{1}{4}(C +$$

232. Find x, when $\tan^{-1} \frac{1}{4} + 2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{6} + \cot^{-1} x = \frac{1}{2}\pi$.

233. A, B, C, are three stations. From B a base BD (a) is measured along BA: also the angles CBD, BDC, BCA, (θ, ϕ, ψ) are observed, as well as the depression (α) of A and the elevation (β) of B with respect to C. Find the height of C above the horizontal plane through A.

234. If, in any plane triangle, lines (l, m, n) be drawn bisecting the angles A, B, C, respectively, then

$$l^2\left(\frac{1}{c}-\frac{1}{b}\right)+m^2\left(\frac{1}{a}-\frac{1}{c}\right)+n^2\left(\frac{1}{b}-\frac{1}{a}\right)=0.$$

235. Shew that $\sqrt{(1+\sin\theta)}=1+2\sin\frac{1}{4}\theta\sqrt{(1-\sin\frac{1}{2}\theta)}$.

236. Find x, when $\cos(x+\frac{3}{2})\alpha+\cos(x+\frac{1}{2})\alpha=\sin\alpha$.

237. If in any triangle the angles A, B, C, be in A. P., shew that $\cot\frac{1}{2}A$, $\cot\frac{1}{2}B$, $\cot\frac{1}{2}C$, will be also in A. P.

238. Shew that $\cos^{-1}\sqrt{\frac{2}{3}}-\cos^{-1}\dfrac{\sqrt{6}+1}{2\sqrt{3}}=\dfrac{\pi}{6}$.

239. Given the perimeter and the three angles of a triangle, determine one of its sides.

240. If $\tan\theta=\cos\alpha\tan\phi$, shew that $\tan(\phi-\theta)=\dfrac{\tan^2\frac{1}{2}\alpha\sin 2\phi}{1+\tan^2\frac{1}{2}\alpha\cos 2\phi}$.

241. A man ascends a mountain by a path, which is the shortest distance between the base and vertex. The inclination of his path to the horizon is at first α; but afterwards it increases suddenly to β, and then continues unaltered. At the top he finds by the barometer that he has ascended *m feet* in vertical height, and observes the angle of depression (γ) of his starting-place. Find the distance travelled by him in the ascent.

242. Shew that in any triangle the length of the line from C, bisecting the side c, is $\frac{1}{2}\sqrt{\{2(a^2+b^2)-c^2\}}$.

243. Find x in the equation $\sec x=2\sin^2x+\cos x$.

244. Prove that $\dfrac{1+(\operatorname{cosec}\alpha\tan x)^2}{1+(\operatorname{cosec}\beta\tan x)^2}=\dfrac{1+(\cot\alpha\sin x)^2}{1+(\cot\beta\sin x)^2}$.

245. Shew that $\cos^{-1}\dfrac{1-m^2}{1+m^2}-\cos^{-1}\dfrac{1-n^2}{1+n^2}=\tan^{-1}\dfrac{2(m-n)(1+mn)}{(1+mn)^2-(m-n)^2}$.

246. Given α, β, γ, the distances from the angles of the point at which the sides of a plane triangle subtend equal angles: find the side c, and the area of the triangle.

247. Shew that $\tan^{-1}\dfrac{x\cos\phi}{1-\sin\phi}-\tan^{-1}\dfrac{x-\sin\phi}{\cos\phi}=\phi$.

248. Find the general value of A in the following equations: (i) $\sin A = \pm\frac{1}{2}\sqrt{2}$; (ii) $\cos A = \pm\frac{1}{2}$; (iii) $\cos A + \sin A =$

249. In the figure of *Euc.* IV. 10, shew that the areas of the circles are as $5+\sqrt{5} : 2$.

250. Shew that the equation $\cos(a+x) = \cos a \sin x + \sin b$ may reduced to the two equations
$\cos(\phi+x) = \sin b \sec a \cos\phi$, $\tan\phi = \sin(45° + a) \sec 45°$ se

251. Solve the equation $\tan\theta + \cot\theta = 4$; and express the val of θ by a single formula.

252. Prove that $\cot^{-1}n - \cot^{-1}(n+1) = \cot^{-1}(1+n+n^2)$; and the sum $\cot^{-1}(1+1+1^2) + \cot^{-1}(1+2+2^2) +$ &c. to n terms.

253. Straight lines m, n, are drawn from the acute angles (A, of a right-angled triangle to the bisections (D, E) of opposite sides. Express the tangents of these angles terms of m and n.

254. Prove that $\dfrac{\sin 3A + \cos 3A}{\sin 3A - \cos 3A} = \dfrac{2 \sin 2A + 1}{2 \sin 2A - 1} \times \tan(45° - A)$.

255. The angles of a triangle form a G. P., whose ratio is $\frac{1}{2}$: sh that the greatest side $= 2$(perimeter) $\sin 12° \, 51' \, 25\frac{5}{7}''$.

256. If $\sin^2 A + \sin^2 B = \sin^2 C$, is the triangle right-angled or n

257. If in the three edges, which meet in one angle of a c three points A, B, C, be taken at distances a, b, c, from angles respectively, then the area of the triangle A

$$= \tfrac{1}{2}\sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}.$$

258. Shew from *Euc.* XI. 21, that there can be only five reg solids, and that in these the faces will be either equilat triangles, squares, or regular pentagons.

259. In a plane triangle, if $a + b = m$, shew that $a = m$ co $b = m \sin^2\phi$, where ϕ may be found from the equation

$$m \sin 2\phi = \sqrt{\{(m+c)(m-c)\}} \sec\tfrac{1}{2}C.$$

260. Find the value of $\cos^{-1}\frac{1}{2}(-1)^m$, m being any integer; express the different results by one single formula.

261. If p, q, r, be the lines (AD, BE, CF) drawn from angles of a triangle to bisect the sides a, b, c, respecti express a side (c) in terms of p, q, r.

262. If $\tan A = \sin 2A$, find A and chord A.

263. Employ a subsidiary angle to adapt $\sqrt{(a+b)} + \sqrt{(a-b}$ logarithmic computation.

264. Shew that the length of the line CD, which is drawn to the side AB of a triangle, so as to bisect the angle C, is

$$\frac{2ab}{a+b}\cos\frac{C}{2}.$$

265. A, B, C, are three points in a horizontal plane, at known distances a, b, c, from each other, at each of which the height of a tower, whose foot is in the same plane, is seen under the same angle (α). Determine it.

266. Shew that the equation $\sin\theta + \sin\phi = \sin(\theta+\phi)$ can only hold when one of the quantities, θ, ϕ, or $\theta+\phi$, is a multiple of 360°, (including *zero*).

267. In a right-angled triangle, lines CD, CE, are drawn from the right-angle C, making towards the same parts angles α, β, with the hypothenuse: shew that the area of the triangle CDE is $\frac{a^2b^2}{2c^2}(\cot\alpha \sim \cot\beta)$.

268. If $ABCD$ be a quadrilateral, whose opposite angles are supplementary, shew that $\tan\frac{A}{2} = \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}$; and shew how this may be reduced to the corresponding expression for a triangle, by supposing one of the sides (b, c) opposite to the angle A to vanish.

269. Prove that $\log_e\left(\frac{\cot\frac{1}{2}\theta}{\sin\theta}\right) = \frac{\cos\theta}{1} + \frac{\cos^2\theta}{2} + \frac{\cos^3\theta}{3} + \&\text{c.}$

270. A triangular field is to be divided among three persons in proportion to their ages, which are m, n, p, years respectively, by means of two fences, parallel to one of its sides. Shew how this may be done.

271. Prove the formulæ

$$\frac{\operatorname{cosec}2A}{1+\operatorname{cosec}2A} = \frac{1+\tan^2 A}{(1+\tan A)^2} \text{ and } \operatorname{cosec}A = \cot\tfrac{1}{2}A - \cot A.$$

272. Sum *ad infinitum* $(1-3^{-\frac{1}{2}}) - \frac{1}{3}(1-3^{-\frac{3}{2}}) + \frac{1}{5}(1-3^{-\frac{5}{2}}) - \&\text{c.}$

273. Prove, *a priori*, that $\sin mA$, when expressed in terms of $\sin A$, will have one or two values, according as m is odd or even, and that $\cos mA$, in terms of $\cos A$, will have only one value, m being in each case a positive integer.

274. Find θ when $2 + \cot^2\theta = 3\sec^4\theta - \tan^2\theta$.

275. When $x = \frac{1}{2}\pi$, shew that $\left(x - \frac{2x^2}{\pi}\right)\tan x = 1$.

276. A ladder, whose length is 30 feet, stands against a wall, at an angle of 60° with the horizon. At what distance from its top must another of the same length be fastened, at an angle of 75° with the horizon, so as just to reach a window 48 feet from the ground?

277. Solve the equation $8^x . 125^{2-x} = 2^{4x+3} . 5^x$, given $\log 2 = .30103$.

278. On the sides of an equilateral triangle three squares are described. Compare the area of the triangle formed by joining the centres of these squares with the area of the equilateral triangle.

279. From the values of BD and $\sin A$ in (116), obtain the expression for the radius of the circle, which circumscribes a given quadrilateral, in terms of its sides.

280. If $\cos 2\theta = 4 \cos a \cos \theta$, shew that $\cos\theta = \cos 45^\circ \tan(\phi \pm 45^\circ)$, where $\phi = \frac{1}{2}\tan^{-1}(\cos a \sec 45^\circ)$.

281. Let r_1, r_2, r_3, be the radii of the circles, which touch one side of a triangle ABC and the other two sides produced: then $r_1 \cot\frac{1}{2}A = r_2 \cot\frac{1}{2}B = r_3 \cot\frac{1}{2}C = \sqrt{(r_1r_2 + r_1r_3 + r_2r_3)}$.

282. Shew that the ratio of the areas of regular decagons, described in and about the same circle, will be as 7 : 8 nearly, more nearly as 29 : 32, and still more nearly as 123 : 136.

283. Determine $\cos\theta$ and $\cos\phi$ from the equations

$$p \sin^4\theta - q \sin^4\phi = p,\quad p \cos^4\theta - q \cos^4\phi = q.$$

284. From the top of a tower a person observes the depressions (a, β) of two distant points in the horizontal plane at the foot of the tower, (whose distance from each other he knows to be a miles,) and also the angle (θ) subtended at his eye by the line joining the two points. Find the height of the tower, and adapt the expression to logarithmic computation.

285. If the quadrilateral in (116) can also have a circle inscribed within it, shew that the sums of its opposite sides are equal, and thence that its area $= \sqrt{(abcd)}$: determine also the radius of the circle.

286. Shew that $\sin\frac{1}{2}A$, in terms of $\cos A$, will have two values, but in terms of $\sin A$, four values.

287. Calculate the common log of 255 to four places of decimals, having given $\log_{10} 2 = .30103$, and the *modulus* .4342944819.

288. *Given* $\sin(a - \theta) = \cos(a + \theta)$, find θ.

9. If $\tan\theta = \dfrac{\sin\phi}{\sin\psi}\sin(\phi+\psi)$, shew that

$$\tan\phi = \frac{\sin\theta}{\sin(\theta+\phi+\psi)}\sin(\phi+\psi).$$

0. In a given circle is inscribed an equilateral triangle, and in that triangle another circle, and so on *ad inf.*: shew that the sum of the circumferences of all the inner circles is equal to that of the given circle, and the sum of their areas to a third of its area.

1. Find θ from the equation $\sin\theta - \cos\theta = 4\cos^2\theta\sin\theta$.

2. The angles A, B, C, of a triangle are as the numbers 2, 3, 4: shew that $\cos\frac{1}{2}A = \dfrac{a+c}{2b}$.

3. Three circles, whose radii are a, b, c, touch each other externally: prove that the tangents at the point of contact meet in a point whose distance from any one of them is $\left(\dfrac{abc}{a+b+c}\right)^{\frac{1}{2}}$.

4. If a circle be inscribed in a square, and between it and the four angular points four other circles be described, and so on continually, shew that the sum of the perimeters of all these circles : that of given circle :: $2\sqrt{2}-2 : 1$, and the corresponding areas :: $3\sqrt{2}-4 : 2$.

5. Shew that the sum of the squares of the distances of the centre of the inscribed circle from the angular points of a triangle $= ac + bc + ab - \dfrac{6abc}{a+b+c}$.

6. Adapt the expression $\dfrac{1}{a + b\cos x}$ to logarithmic computation, both when a is greater and less than b.

7. Prove that $\frac{2}{3}(8\sin\frac{1}{4}\theta - \sin\frac{1}{2}\theta) = \theta$ nearly, if θ be small.

8. AB, AC, are the chords of arcs of 60° and 90° in a circle, whose centre is O: shew that, if AB and OC be produced to meet in D, the arc, whose chord is equal to DC, has its cosine and chord equal.

9. Solve the equation $\tan^{-1}\dfrac{4(x-x^3)}{1+x^2} = 2\pi - 2\,\text{chd}^{-1}\dfrac{2}{\sqrt{(1+x^2)}}$.

0. A person, walking from C to D on the horizontal road, can see plainly an obelisk on the summit of a hill at A, at every *point but E*, where he can *just* see the obelisk over

the hill B. He then measures a base EC (a), and at C observes the elevations (α, β) of A and B, as well as the angles ACB, ACE, (γ, δ). At E he observes the angle AEC (ϵ). Find the heights of the two hills.

301. From the four angles of an inscribed quadrilateral, lines are drawn to the diagonals, making equal angles with them: shew that the product of two of these lines, taken alternately, is equal to the product of the other two.

302. If $\theta + \phi + \psi = 90°$, shew that

$$(\tan\theta + \tan\phi + \tan\psi)(\cot\theta + \cot\phi + \cot\psi) = 1 + \operatorname{cosec}\theta \operatorname{cosec}\phi \operatorname{cosec}\psi.$$

303. Find θ from the equation $\sin^2 2\theta - \sin^2\theta = \frac{1}{4}$; and express its value by a single formula.

304. Find the circular measure of $5° \, 43' \, 46''.46$; and by means of the formula $\sin a = a - \frac{1}{6}a^3 + \&c$, obtain the numerical value of the sine of this angle, to seven places of decimals.

305. If $\sin\frac{1}{3}A$ be found from a given value of $\cot A$, shew how many values of $\sin\frac{1}{3}A$ may be expected.

306. Find x in the equation $\sin^{-1}(2x) - \sin^{-1}(x\sqrt{3}) = \sin^{-1}x$.

307. If $A + B + C = \pi$, shew that

$$\frac{\tan A}{\tan B} + \frac{\tan B}{\tan C} + \frac{\tan C}{\tan A} + \frac{\tan A}{\tan C} + \frac{\tan C}{\tan B} + \frac{\tan B}{\tan A}$$
$$= \sec A \sec A \sec C - 2.$$

308. P is any point in the circumference of a circle, whose diameter is AB. Take $\angle PAB = \theta$, and find, geometrically, the expression for $2\sin\theta$ in terms of $\sin 2\theta$.

309. If $\dfrac{2}{1+x} = \dfrac{\sin\beta\sin\theta}{\cos(\beta-\theta)} = \dfrac{\tan(\theta - a)}{\cot\beta}$, shew that

$$x = \{\cot\tfrac{1}{2}a - 2\tan\beta\}^{\frac{1}{2}}\{\cot\tfrac{1}{2}a + 2\tan\beta)^{\frac{1}{2}}.$$

310. Write down the first twenty positive values of θ, which satisfy the equation $\dfrac{\sin 2\theta}{\theta}\,\dfrac{\sin 12\theta}{\sin 3\theta} = 0$; and express generally the values of θ which satisfy it.

311. If two lines, at right angles to one another on an inclined stratum, make angles a, β, with the horizontal plane, shew that the dip (θ) of the stratum is given by the equation

$$\cos\theta = \{\cos(a+\beta)\cos(a-\beta)\}^{\frac{1}{2}}.$$

312. A circle revolves in its own plane round a point O, and from a point P in that plane the greatest and least angles under which the circle is seen are $2a$, 2β. Shew that the distance *of O from* the centre of the circle = $PO \sin\phi$, where

$$\tan^2(45° + \tfrac{1}{2}\phi) = \sin a \div \sin\beta.$$

3. At what distance may the Peak of Teneriffe, which is $2\frac{1}{2}$ miles above the level of the sea, be first seen from the top-mast of a ship, 80 feet above the surface?

4. Given the areas A, B, of the regular inscribed and circumscribed polygons of a circle, each of the same number of sides, find the areas of the corresponding regular polygons of twice the same number of sides.

5. The sides of a triangle are in A. P., and its area is to that of an equilateral triangle of the same perimeter as $3 : 5$. Shew that its largest angle is 120°.

6. If the sides of a triangle are a, b, c, and another triangle be formed, whose sides are proportional to $a \cos A$, $b \cos B$. $c \cos C$, shew that the angles of this latter triangle will be the supplements of $2A$, $2B$, $2C$, respectively.

7. Eliminate θ and ϕ from the equations

$$a \sin^2\theta + b \cos^2\theta = m,\quad a \cos^2\phi + b \sin^2\phi = n,\quad a \tan\theta = b \tan\phi.$$

8. A staff, one foot long, stands at the top of a tower 200 *feet* high. Shew that the angle which it subtends at a point in the horizontal plane, 100 *feet* from the base of the tower, is 6′ 51″, nearly.

9. Shew that in any triangle $\tan(B + \frac{1}{2}A) = \frac{c+b}{c-b} \tan\frac{1}{2}A$; and if $3c = 7b$, and $A = 6° 37' 24''$, find the other angles, given

$$\log 2 = .30103,\quad \text{L}\tan 3° 18' 42'' = 8.7624080,$$
$$\text{L}\tan 8° 14' = 9.1604569,\quad \text{L}\tan 8° 13' 50'' = 9.1603083.$$

0. Given $\tan\theta + \tan\phi + \tan\psi = 1 + \frac{4}{3}\sqrt{3}$, $\tan\theta \tan\phi \tan\psi = 1$, and $\tan\theta \tan\phi + \tan\theta \tan\psi + \tan\phi \tan\psi = 1 + \frac{4}{3}\sqrt{3}$; find θ, ϕ, ψ.

1. Given $\tan\frac{1}{2}a = \tan^3\frac{1}{2}\beta$, and $\tan\beta = 2\tan\phi$, shew that ϕ is an arithmetic mean between a and β.

2. Two objects are seen in the same direction to the SE, and, when the observer has moved 8 miles due S, the one bears NE, and the other due E: how far are they apart?

3. If the tangents of the angles, A, B, C, of a triangle increase in a G. P., whose ratio is n, shew that $\sin 2A = n \sin 2C$.

4. Find x from the equation $\sec^{-1}a + \sec^{-1}\frac{x}{a} = \sec^{-1}b + \sec^{-1}\frac{x}{b}$.

325. If in a regular polygon lines be drawn from the extremities of any side (a) to those of any other side, so as to cross each other, shew that their intersections will all lie in a circle whose radius is $\frac{a}{2} \operatorname{cosec} \frac{2\pi}{n}$.

326. In the ambiguous case of oblique triangles, if c, c', be the two values found for the third side of the triangle, shew that

$$c^2 + 2cc' \cos 2B + c'^2 = 4b^2 \cos^2 B.$$

327. If r, r_1, r_2, r_3, be the radii of the inscribed and escribed circles of a triangle, and R of its circumscribing circle, shew that the area of the triangle $= \sqrt{(rr_1r_2r_3)}$, and that

$$r_1 + r_2 + r_3 - r = 4R.$$

328. Shew that the triangle, square, and hexagon, are the only regular figures by which space can be completely filled up about a point.

329. Given $\sin(\theta+\phi) = \tan(\theta-\phi) = \frac{1}{2}(\tan\theta - \tan\phi)$, find θ and ϕ.

330. Find x from the equation

$$\operatorname{vers}^{-1}(1+x) - \operatorname{vers}^{-1}(1-x) = \tan^{-1} 2\sqrt{(1-x^2)}.$$

331. Given that the line, bisecting the angle A (also given) of a triangle, divides the base in the ratio of $m : n$, find $\tan B$, and adapt the result to logarithmic computation.

332. If the angle C of a triangle be very obtuse, shew that its defect from $180^\circ = 206265 \sqrt{\frac{2(a+b)(a+b-c)}{ab}}$ seconds, nearly.

333. Two objects, P, Q, are observed from a ship to be at the same instant in a line bearing N. 15° E. The ship sails NW for 5 miles, and then P bears due E, and Q bears NE. Find the distance between P and Q.

334. The perimeter of a triangle is four times its base (c): shew that if r be the distance of the centre of its inscribed circle from one end of the base, and ϕ the angle it makes with the base, then $r(1+\sin^2\phi) = c\cos\phi$.

335. If $r = \tan\{\sin^{-1}(\frac{\sin z}{1+n})\}$, where n is very small, prove that $r = \tan\{z(1 - n\sec^2 z)\}$, nearly.

336. The distance between the centres of two wheels is a, and the sum of their radii is c: shew that the length of the string, which crosses between the wheels and just wraps around *them, is* $2\{\sqrt{(a^2-c^2)} + c\cos^{-1}(-\frac{c}{a})\}$.

7. Given $a = \sec\theta + \operatorname{cosec}\theta\tan^3\theta(\operatorname{cosec}^2\theta + 1)$

$$b = \tan\theta - \tan^3\theta(\operatorname{cosec}^2\theta + 1);$$

shew that $a^{\frac{2}{3}} - b^{\frac{2}{3}} = 2^{\frac{2}{3}}$.

8. A person standing at known distances (a, b) from two towers, in the same vertical plane with himself, observes their apparent altitudes to be the same. He then walks a given distance (c) towards the towers, till the elevation of the one is double that of the other. Find their heights.

9. The alternate angles of a regular pentagon being joined, shew that the area of the interior pentagon thus formed is to that of the original figure as $3 + \sqrt{5} : 3 - \sqrt{5}$.

0. In the figure of *Euc.* I. 1, find the area of the part common to both the circles, the radius of each being a.

1. A piece of paper in the form of a circular sector, whose angle is 2 and radius 2 inches, is formed into a conical cap. Find the area of the conical surface, and also of the base of the cone.

2. Shew that x cannot be real in the equation $\tan^3 x = \tan(x - a)$, if $\sin a$ be greater than $\frac{1}{3}$.

3. Given $\tan^2 x + \sec 2x = 7 - \frac{10}{3}\sqrt{3}$, find x.

4. From the extremity A of the diameter ACB of a semicircle, whose centre is C, a line AP is drawn to the circumference, bisecting the semicircle. If θ be the circular measure of the complement of PCB, prove that $\cos\theta = \theta$.

5. Two circles have a common radius r, and a circle is described touching this radius and the two circles: prove that the radius of the circle which touches the three $= \frac{3}{66}r$.

6. Determine x and y from the equations

$$\tan^{-1}x - \tan^{-1}y = \cot^{-1}2y - \cot^{-1}2x = \tfrac{1}{4}\pi.$$

7. Shew that $\frac{1}{3}\{2(\sin 2\theta - \sin\theta) + (\tan 2\theta - \tan\theta)\} = \theta$, very nearly, if θ be small.

8. A ship, sailing SSW, was observed by another, sailing away from the former, due S, to subtend an angle of 20′: its length being known to be 150 feet, determine its distance.

9. The sides AB, BC, &c. of an inscribed quadrilateral are in G. P., of which the common ratio is m: shew that

$$\tan ABC : \tan BCD :: m^2 - 1 : m^2 + 1.$$

0. If $\tan a \tan x = \tan^2(a + x) - \tan^2(a - x)$, find $\cos x$.

351. If two semicircles be described upon the bounding radii of a quadrant, the circle which touches the three circumferences will have its radius : radius of quadrant :: $\sqrt{2}-1 : 3\sqrt{2}-1$.

352. A circle is inscribed in a triangle, and a second triangle is formed, whose sides are equal to the distances of the points of contact from the angles of the triangle. If r be the radius of the circle inscribed in the first triangle, and r', r'', the radii of the inscribed and circumscribed circles of the second triangle, then will $r^2 = 2r'r''$.

353. The area of a regular pentagon : area of isosceles triangle used in its construction (*Euc.* IV. 11) :: $\sqrt{5} : 1$.

354. Determine x, when

$$2\operatorname{cosec}^{-1}\sqrt{(x^2-2x+2)} = \tfrac{1}{6}\pi + \operatorname{chd}^{-1}\frac{2}{\sqrt{(x^2+2x+2)}}.$$

355. A hexagon is formed by drawing lines from each angle of a triangle, perpendicular to the sides containing that angle. Shew that a circle may be described about it, and that the area of the hexagon is double of that of the triangle.

356. Shew that in the ambiguous case, when a and b are very nearly equal, the number of seconds in the angle C will be (nearly) $\dfrac{(a-b)\tan B}{b\sin 1''}$ or $(\pi-2B)\times 60\times 60 - \dfrac{(a-b)\tan B}{b\sin 1''}$.

357. Solve the equations

$$\theta-\phi = \sin^{-1}\frac{1}{\sqrt{10}},\quad \frac{\sin^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi}{\sin^2\theta\cos^2\phi-\cos^2\theta\sin^2\phi} = \frac{5}{3}.$$

358. In any triangle, given the sides b, c, shew that if A receive a small increment (a), then B and C will receive small decrements, respectively,

$$a\sin B\cos C\operatorname{cosec} A,\ \text{and}\ a\sin C\cos B\operatorname{cosec} A.$$

359. Prove geometrically the expressions for $\sin\theta$ in terms of $\cos 2\theta$ and $\sin 2\theta$; and explain the two values in the former case, and the four in the latter.

360. The angles of a triangle are as 3, 4, 5: find its area, having given r, the radius of its inscribed circle.

361. Given

$$\sin\theta\sqrt{(1+\tan^2\alpha\tan^2\beta)}+\cos\theta\sqrt{(1-\tan^2\alpha\tan^2\beta)} = \tan\alpha+\tan\beta,$$

find θ in a form adapted to logarithmic computation.

362. If the middle points of the sides of a triangle be joined with the opposite angles, and r_1, r_2, &c. R_1, R_2, &c. be

the radii of the circles described *in* and *about* the six triangles thus formed, shew that

$$R_1R_3R_5 = R_2R_4R_6 \text{ and } \frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6}.$$

3, If from the ends of the base of a semicircle two chords be drawn, crossing each other and making angles α, β, with the base, and if the extremities of these chords be joined, the areas of the two triangles thus formed will be as $\cos^2(\alpha+\beta) : 1$.

4. Shew that $\sin(45^\circ \pm x) = \sqrt{\{\tfrac{1}{2}(1 \pm \sin 2x)\}}$; and thence prove that $2\cos(60^\circ \pm \frac{30^\circ}{2^n}) = \sqrt{[2 - \sqrt{\{2 - \sqrt{(2 - \&c)}\}}]}$, where the (2) occurs n times, and the upper or lower sign is to be used, according as n is even or odd.

5. If θ be a small angle, containing n'', shew that

$$\log_{10} n = \text{L} \sin n'' + \tfrac{1}{3}(10 - \text{L} \cos\theta) - \text{L} \sin 1'', \text{ nearly.}$$

6. Eliminate θ from the equations

$$a \tan\theta + b \sec\theta = c, \quad a \cot\theta + b \operatorname{cosec}\theta = d.$$

7. If, in any triangle *ABC*, a line be drawn from *A* to the point *D*, where the inscribed circle touches the opposite side, and if in the two triangles, thus formed, circles be inscribed, shew that they will touch *AD* in the same point.

8. If $\theta + \phi + \psi = 90^\circ$, shew that $\sin\theta + \sin\phi + \sin\psi - 1 = \sqrt{2}(\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta)(\cos\tfrac{1}{2}\phi - \sin\tfrac{1}{2}\phi)(\cos\tfrac{1}{2}\psi - \sin\tfrac{1}{2}\psi)$.

9. The distance between two points is c, and their distances from a given line a, b. Of all triangles that can be formed, having the same base c, and with their vertices in the given line, the area of that which has the least perimeter is

$$\frac{ab}{a+b}\sqrt{\{(c+a-b)(c+b-a)\}}.$$

0. From a vessel sailing SW at the rate of 8 miles an hour a landmark is seen to bear SE by E, and 1½ hour afterwards it bears E by N. Determine the distance of the vessel from the landmark at each observation, given

$$\sin 33^\circ\, 45' = .555, \quad \cos 11^\circ\, 15' = .980.$$

1. If from any point *P*, within or without a triangle, perpendiculars *Pa*, *Pb*, *Pc*, be dropped upon the sides *BC*, *AC*, *AB*, and circles be described about the triangles *Pab*, *Pac*, *Pbc*, the area of the triangle, formed by joining the centres of these *circles, will* be ¼ of the area of the triangle ABC.

372. Find the values of x which satisfy the equation

$$\sin\{2\cos^{-1}\cot(2\tan^{-1}x)\} = 0.$$

373. If $a = b + c$, shew that

$$\left\{\frac{1+\cos ma}{ab} - \frac{1-\cos mc}{bc}\right\}^2 + \left\{\frac{\sin ma}{ab} - \frac{\sin mc}{bc}\right\}^2$$

$$= \frac{1}{abc}\left\{\frac{a^2+b^2+c^2}{abc} + 2\,\frac{\cos ma}{a} - 2\,\frac{\cos mb}{b} - 2\,\frac{\cos mc}{c}\right\}:$$

and find the value of either side, when $a = c$.

374. If θ be the angle between the diagonals of a parallelogram, whose sides (a, b) are inclined at an angle (α), find $\tan\theta$.

375. Six circles are described between the escribed circles of a triangle and the angular points, each touching a side and a side produced. Shew that the products of their radii, taken alternately, are each equal to $Ss\tan^2\frac{1}{4}A\tan^2\frac{1}{4}B\tan^2\frac{1}{4}C$, where S is the area of the triangle, and $2s = a + b + c$.

376. If $\cos^2\theta = \dfrac{\cos\alpha}{\cos\beta}$, $\cos^2\theta' = \dfrac{\cos\alpha'}{\cos\beta}$, and $\dfrac{\tan\theta}{\tan\theta'} = \dfrac{\tan\alpha}{\tan\alpha'}$, shew that $\tan\frac{1}{2}\alpha\tan\frac{1}{2}\alpha' = \tan\frac{1}{2}\beta$.

377. Shew that

$$\tfrac{1}{2}a^3\operatorname{cosec}^2\left(\tfrac{1}{2}\tan^{-1}\frac{a}{b}\right) + \tfrac{1}{2}b^3\sec^2\left(\tfrac{1}{2}\tan^{-1}\frac{b}{a}\right) = (a+b)(a^2+b^2).$$

378. Prove that $\tan\frac{1}{4}A$, when expressed in terms of $\sin A$, will have four different values.

379. If $\cot\theta - \cot\alpha = 2\sec 2\phi$, and $\cot\theta' - 3\sin\alpha' = \sin^2\phi$, find what values must be given to α and α', so that $\tan\theta - \tan\theta'$ may have a fixed numerical value for all values of ϕ.

380. If a, a', be homologous sides of similar triangles, described *in* and *about* a circle, shew that $a = 4a'\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$.

381. The elevations (α, β, γ) of a balloon are taken at the same moment, at three points A, B, C, in the same horizontal line. Given $AB = a$, $BC = b$, determine its altitude.

382. Given $\operatorname{vers}^{-1}\dfrac{x}{a} - \operatorname{vers}^{-1}\dfrac{bx}{a} = \operatorname{vers}^{-1}(1-b)$: find x.

383. A person, wishing to determine the length of a wall upon the opposite side of a stream, places himself due South of one end, and then due West of the other, at such distances that the angle which the wall subtends at each position is (α). Shew that, if a be the distance between the two stations, the length of the wall will be $a\tan\alpha$.

4. If $\cos(a-x)=\cos a\cos b$, where x is very small, prove that
$$x=-2\cot a\sin^2\tfrac{1}{2}b(1-\cot^2 a\sin^2\tfrac{1}{2}b),\text{ very nearly.}$$
How may we find from this the number of seconds in x?

5. If $m\cos\phi=\sqrt{\{\frac{1}{3}(1-m^2)\}}$ and $\tan a=\tan^3\frac{1}{2}\phi$, shew that
$$m=\tfrac{1}{2}(\cos^{\frac{2}{3}}a+\sin^{\frac{2}{3}}a)^{\frac{3}{2}}.$$

6. Two ships are sailing uniformly, with velocities u, v, along lines inclined at an angle θ. Given a, b, their distances at the same moment from the point of intersection of their courses, find their least distance from each other.

7. On the sides of a triangle ABC, as diameters, three circles are described intersecting in P, Q, R: shew that the perimeter of ABC : that of PQR :: $1:4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$.

8. Find the value of x, when $\sin 2x=\sin^2 3x$.

9. A and B are two observed angles of a plane triangle, which have small unknown errors. What is the form of the triangle which will give $\sin(A+B)$ with the least error?

0. Shew that θ may be found from the equation
$$\sin^3\theta=\sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta),$$
where $\alpha+\beta+\gamma=\pi$, by either of the formulæ
$$\cot\theta=\cot\alpha+\cot\beta+\cot\gamma,\text{ or }\operatorname{cosec}^2\theta=\operatorname{cosec}^2\alpha+\operatorname{cosec}^2\beta+\operatorname{cosec}^2\gamma.$$

1. A circle is described about a triangle ABC, and a new triangle $A'B'C'$ is formed by joining the points of bisection of the arcs, subtended by the sides of ABC. Shew that the sides of this triangle are $\frac{1}{2}a\operatorname{cosec}\frac{1}{2}A$, $\frac{1}{2}b\operatorname{cosec}\frac{1}{2}B$, $\frac{1}{2}c\operatorname{cosec}\frac{1}{2}C$, and that its area : area ABC :: $1:8\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$.

2. The shadows of two vertical walls, which are at right angles to each other, and are a and a' feet in height, are observed, when the Sun is due South, to be b and b' feet in breadth. Shew that, if α be the Sun's altitude, and β the inclination of the first wall to the NS line (or *meridian*,) then
$$\cot\alpha=\sqrt{\left(\frac{b^2}{a^2}+\frac{b'^2}{a'^2}\right)},\quad\cot\beta=\frac{ab'}{a'b}.$$

3. The circles described through any three of the centres of the four circles, that can be drawn touching the sides or sides produced of a triangle, have all the same radius, which is twice that of the circumscribing circle of the triangle.

4. If P be the centre of a circle, radius R, which passes through *the three* angles of a triangle, and Q the centre

of a circle, radius r, which touches the three sides, then $PQ = \sqrt{(R^2 \mp 2Rr)}$, $\mp$ according as the latter circle lies *within* or *without* the triangle.

395. If Q_1, Q_2, Q_3, Q_4, be the centres of the inscribed and escribed circles of a triangle, and P that of the circumscribing circle, shew that $PQ_1^2 + PQ_2^2 + PQ_3^2 + PQ_4^2 = 12R^2$.

396. Prove that $\sin\theta = \theta \cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \cos\frac{\theta}{2^3} \ldots$ *ad infinitum*.

397. Find the value of $\frac{1}{\sqrt{x}} \text{vers}^{-1}\frac{x}{a}$, when $x = 0$.

398. At noon, a column in the ESE cast upon the ground a shadow, the extremity of which was in the direction NE. The elevation of the column, as seen from the spectator, being (α), and his distance from the extremity of the shadow a feet, determine the height of the column.

399. Find the value of $(1 + \sec\theta)(1 + \sec\frac{\theta}{2}) \ldots (1 + \sec\frac{\theta}{2^{n-1}})$.

400. The sides of a triangle are in A. P., and the distance of the centres of the inscribed and circumscribed circles is a mean proportional between the greatest and least of them. Shew that the sides are as the numbers $\sqrt{5} - 1$, $\sqrt{5}$, $\sqrt{5} + 1$.

401. If on the sides of a triangle are described three triangles, whose vertical angles are equal to those angles of the triangle, which are respectively opposite to them, shew that the triangle, formed by joining the centres of the circles circumscribed about them, is in every respect equal to the original one.

402. Find the value of θ, when

$$(1 + \cos\alpha \cos\theta) \tan(\theta + \alpha) + (\cos\alpha + \cos\theta) \sin\alpha = 0.$$

403. If $\alpha < 45°$, prove that

$$2\tan^{-1}\{\tan^{\frac{1}{2}}(45° - \alpha)\tan\tfrac{1}{2}\beta\} = \cos^{-1}\frac{\tan\alpha + \cos\beta}{1 + \tan\alpha\cos\beta}.$$

404. A flag-staff, 8 feet high, on the top of a tower, subtends an angle of 57′ 17″.75 at 100 yards from the foot of the tower: determine its height.

405. If x, y, z, be the lengths of lines, bisecting the angles of a triangle, and terminated by the opposite sides, a, b, c, shew that

$$(b + c)^2\frac{x^2}{bc} + (a + c)^2\frac{y^2}{ac} + (a + b)^2\frac{z^2}{ab} = (a + b + c)^2.$$

6. Find the area between three circles which touch one another, and whose diameters are $\sqrt{3}-1$, $\sqrt{3}+1$, $3-\sqrt{3}$, $(\pi=\frac{22}{7})$.

7. The base of a pyramid being square, and its sides equilateral triangles, if θ, ϕ, be the inclinations of its sides to the base and to each other respectively, then $\cos^2\theta = \cos(\pi-\phi)$.

8. Shew that, if $a+\beta+\gamma = 90°$, then

$$\frac{\cos a + \sin\gamma - \sin\beta}{\cos\beta + \sin\gamma - \sin a} = \frac{1+\tan\frac{1}{2}a}{1+\tan\frac{1}{2}\beta}.$$

9. If the circumference of a circle be divided into arcs of 60° in the points P_1, P_2, &c, and if P_1P_3 be joined, cutting the radius OP_2 in Q_1, and then Q_1P_4 be joined, cutting OP_3 in Q_2, and so on, then OQ_1, OQ_2, OQ_3, &c. are respectively the half, third, fourth, &c. parts of the radius.

0. In a segment of a circle, whose radius is r, and length of arc l, a circle is described, touching the middle point of the arc. Shew that the radius of the circle, which touches the chord and arc of the segment, and also the inscribed circle, is $\frac{r}{4}\sin^2\frac{l}{2r}$.

1. Prove that $\tan^3\frac{1}{4}\{\sin^{-1}(3\sin a)+a\} = \tan\frac{1}{4}\{\sin^{-1}(3\sin a)-3a\}$.

2. Solve the equation $\cos 7\theta + 7\cos\theta = 0$.

3. If A, B, C, be the angles of a triangle, shew that

$$\cos nA + \cos nB + \cos nC = 1 \pm 4\sin\tfrac{1}{2}nA\,\sin\tfrac{1}{2}nB\,\sin\tfrac{1}{2}nC,$$
$$\text{or } = -1 \pm 4\cos\tfrac{1}{2}nA\,\cos\tfrac{1}{2}nB\,\cos\tfrac{1}{2}nC,$$

according as n is odd or even.

4. The centres of the escribed circles of a triangle are joined, so as to form another triangle; this is similarly treated, and so on. Shew that the radii of the circumscribing circles will form a G. P., whose ratio is 2.

5. From the equation $h\cos(\theta+a)+\cos a = 0$ derive the equation

$$(1+h\cos\theta)\tan(\theta+a) + (h+\cos\theta)\,h\tan a = 0.$$

6. Shew that $2\cos\frac{45°}{2^{n-1}} = \sqrt{\{2+\sqrt{(2+\&c)}\}}$ to n terms.

7. Squares are described upon the sides of a triangle ABC, and their centres are joined, forming the triangle LMN: prove that area $LMN = (1+\frac{1}{2}p)$ area ABC,

where $p = \cot A + \cot B + \cot C$.

418. On the side of a hill there are two places, B, C, inaccessible to each other, but known to be at the same distance (a) from the same station A. At the lower place C, the horizontal angle (α) between A and B is measured, as well as their altitudes (λ, μ). Shew that the distance BC

$$= 2a\{\cos(\lambda-\mu)\cos^2\tfrac{1}{2}\alpha - \cos(\lambda+\mu)\sin^2\tfrac{1}{2}\alpha\}.$$

419. If $\tan\theta = \frac{b}{a}$, shew, by Demoivre's Theorem, that

$$(a+b\sqrt{-1})^{\frac{m}{n}} + (a-b\sqrt{-1})^{\frac{m}{n}} = 2(a^2+b^2)^{\frac{m}{2n}}\cos\frac{m}{n}\theta.$$

420. Shew that if $2\cos\alpha_n = x_n + \dfrac{1}{x_n}$, and $\alpha_1 + \alpha_2 + \text{\&c} + \alpha_n = 2\pi$, then $x_1x_2x_3 \ldots x_n = 1$.

421. If $a^2+b^2 = 2$ and $ab = \sqrt{-1}\tan 2\phi$, then $a = \cos\phi\sqrt{(2\sec 2\phi)}$.

422. If $\theta = \cos\theta$, shew that θ contains 42° 20′, nearly.

423. Find the sum of n terms of the series

$$\tan^{-1}a + \tan^{-1}\frac{a}{1+1.2a^2} + \tan^{-1}\frac{a}{1+2.3a^2} + \text{\&c}.$$

424. If $e^x\cos x$ be expanded in a series of ascending powers of x, find the coefficient of x^n.

425. Sum to n terms the series $\sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + \text{\&c}$.

426. Prove that $\sin 5^\circ \sin 15^\circ \sin 25^\circ \ldots \sin 85^\circ = \dfrac{1}{256\sqrt{2}}$.

427. Find by Trigonometry the roots of $x^3 - 3x + 1 = 0$.

428. Prove that the sum *ad infinitum* of the series, whose n^{th} term is $(-1)^{n-1}\dfrac{n}{2n-1}\cdot\dfrac{1}{\underline{|2(n-1)}}$, is $\dfrac{1}{\sqrt{2}}\sin(\dfrac{\pi}{4}+1)$.

429. Given $\theta = \cot\theta$, shew that $\theta = 49^\circ\ 17'$ nearly.

430. Shew that $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$.

431. If l be the length of a curve BC upon a line of railroad, and c the direct distance between its extreme points, shew that the radius AB of the circle of which the curve is a portion is $\dfrac{1}{2}\sqrt{\dfrac{l^3}{6(l-c)}}$, nearly.

432. Find by Trigonometry the roots of $2x^3 - 3x - 1 = 0$.

433. If $\tan\alpha = m\tan\beta$, shew that

$$\alpha = \beta + \frac{m-1}{m+1}\sin 2\beta + \frac{1}{2}\left(\frac{m-1}{m+1}\right)^2\sin 4\beta + \text{\&c}.$$

Shew that $x\sin\theta + \frac{x^2}{2}\sin 2\theta + \frac{x^3}{3}\sin 3\theta + \&c = \tan^{-1}\left(\frac{x\sin\theta}{1 - x\cos\theta}\right)$.

Solve the equation $8x^3 - 36x^2 + 42x - 13 = 0$ by Trigonometry.

If m be any integer, prove that

$$\tan\frac{\pi}{4m}\tan\frac{5\pi}{4m}\tan\frac{9\pi}{4m}\dots\tan\frac{(4m-3)\pi}{4m} = \pm 1.$$

7. Shew that $\cos 2\theta \operatorname{cosec}^2 2\theta = \frac{1}{2}\operatorname{cosec}^2\theta - \operatorname{cosec}^2 2\theta$: and thence sum to n terms the series

$$\cos 2\theta \operatorname{cosec}^2 2\theta + 2\cos 4\theta \operatorname{cosec}^2 4\theta + 4\cos 8\theta \operatorname{cosec}^2 8\theta + \&c.$$

3. If $\tan(\frac{1}{4}a - \phi) = \tan^3\frac{1}{4}a$, shew that

$$\phi = \frac{1}{1.3}\sin a - \frac{1}{2.3^3}\sin 2a + \frac{1}{3.3^3}\sin 3a - \&c.$$

9. Sum to n terms the series $\sin^3\theta + \sin^3 2\theta + \sin^3 3\theta + \&c.$

0. In any triangle shew that $\log_e b - \log_e a$
$(\cos 2A - \cos 2B) + \frac{1}{2}(\cos 4A - \cos 4B) + \frac{1}{3}(\cos 6A - \cos 6B) + \&c.$

1. Sum to n terms the series

$$(\cos a + \sqrt{-1}\sin a) + (\cos a + \sqrt{-1}\sin a)^2 + \&c;$$

and deduce from it the sums of

$$\cos a + \cos 2a + \&c, \text{ and } \sin a + \sin 2a + \&c.$$

2. Sum the series $\cos A + \cos 3A + \cos 5A + \&c.$ to n terms.

3. If $\sin\theta = \sin A \sin(B + \theta)$, prove that

$$\theta = \sin A \sin B + \tfrac{1}{2}\sin^2 A \sin 2B + \tfrac{1}{3}\sin^3 A \sin 3B + \&c.$$

4. Sum the series *ad infinitum*

$$\cos\theta + \frac{\cos\theta}{1}\cos 2\theta + \frac{\cos^2\theta}{1.2}\cos 3\theta + \frac{\cos^3\theta}{1.2.3}\cos 4\theta + \&c.$$

5. Prove that

$$\tan\frac{\pi}{2^{x+1}}\left(2^{x-1} + \tan\frac{\pi}{2^{x+1}} + 2\tan\frac{\pi}{2^x} + \,..\, + 2^{x-2}\tan\frac{\pi}{2^3}\right) = 1.$$

6. Prove that

$$\log_e(1 - n\cos\theta) = 2(\log_e a - m\cos\theta - \tfrac{1}{2}m^2\cos 2\theta + \tfrac{1}{3}m^3\cos 3\theta - \&c),$$

where $m = \frac{\beta}{a}$, a, β, being roots of $2x^2 - 2x\sqrt{(1 + n)} + n = 0$.

7. If $x = \cot z$, shew that $\tan^{-1}(x + h) - \tan^{-1}x$

$$= h\sin z.\sin z - \tfrac{1}{2}h^2\sin^2 z.\sin 2z + \tfrac{1}{3}h^3\sin^3 z.\sin 3z + \&c.$$

8. Sum to n terms the series

$$\frac{\sec\theta\sec 3\theta}{\operatorname{cosec} 2\theta} + \frac{\sec 3\theta\sec 5\theta}{\operatorname{cosec} 4\theta} + \frac{\sec 5\theta\sec 7\theta}{\operatorname{cosec} 6\theta} + \&c.$$

9. If $\tan 2\theta = \cos a \tan 2\phi$, shew that

$$\theta = \cos a\tan\phi - \tfrac{1}{3}\cos 3a\tan^3\phi + \tfrac{1}{5}\cos 5a\tan^5\phi - \&c.$$

450. If from one of the angles of a regular polygon of n sides, lines be drawn to all the other angles, shew that their sum $= \frac{1}{2}a \operatorname{cosec}^2 \frac{\pi}{2n}$, a being the length of a side.

451. Expand $\log_e(a\cos^2\theta + b\sin^2\theta)$ in a series of the form

$$A + B\cos^2\theta + C\cos^2 2\theta + D\cos^2 3\theta + \&c.$$

452. The circumference of the inner of two concentric circles, radii R, r, is divided into n equal parts. Shew that the sum of the squares of the lines drawn from the points of division to any point in the outer circumference $= n(R^2 + r^2)$.

453. Prove that

$$\sin\theta \sin\theta - \tfrac{1}{2}\sin 2\theta \sin^2\theta + \tfrac{1}{3}\sin 3\theta \sin^3\theta - \&c = \cot^{-1}(1 + \cot\theta + \cot^2\theta).$$

454. If n be an even integer, shew that

$$\operatorname{cosec}^2\frac{\pi}{2n} + \operatorname{cosec}^2\frac{3\pi}{2n} + \&c + \operatorname{cosec}^2\frac{(n-1)\pi}{2n} = \frac{n^2}{2}.$$

455. If C be the middle point of a semicircular arc whose diameter is AB, and the angle ACB be divided into $n+1$ equal angles by lines terminated by the diameter; shew that the sum of the squares of the reciprocals of the dividing lines $= \frac{1}{2r^2}\{n + \cot\frac{\pi}{2(n+1)}\}$, where r is the radius of the circle.

456. The circumference of a circle, whose centre is C and radius a, is divided into n parts, each of which subtends the same angle at a point O within the circle. If $CO = b$, and r_1, r_2, &c. be the lines from O to the points of division, shew that

$$r_1 + r_2 + \&c = (a^2 - b^2)\left(\frac{1}{r_1} + \frac{1}{r_2} + \&c\right).$$

457. Circles are inscribed in triangles, whose bases are the sides of a regular polygon of n sides, and whose vertices lie in one of the angular points: shew that the sum of the areas of the circles

$$= 16\pi r^2 \sin^2\frac{\pi}{2n}\left\{\frac{n}{4}\sin^2\frac{\pi}{2n} + \frac{n-4}{8}\right\},$$

r being the radius of the circle circumscribing the polygon.

ANSWERS TO THE EXAMPLES.

2. *a*. 4. $\dfrac{n \pm 2p}{m \pm p}$. 6. $\frac{1}{2}ab$. 9. 30 *sq.yds.* 10. .00000042366.

3. $\frac{1}{4}\{\cos(m+n+r)\,\theta + \cos(m+n-r)\,\theta + \cos(m+r-n)\,\theta$
$+ \cos(n+r-m)\,\theta\}$. 17. $\pm \dfrac{\sqrt{5} \pm 1}{2\sqrt{2}}$. 18. 93.97 *ft.*

9. $\sin A = \frac{1}{2}\{\sqrt{(1-\sin 2A)} - \sqrt{(1+\sin 2A)}\}$,
$\cos A = -\frac{1}{2}\{\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}\}$.

5. $-\frac{1}{4}(\sqrt{6}+\sqrt{2})$, $\sqrt{6}+\sqrt{2}$, $-(2+\sqrt{3})$.

1. 30° or $(3n+1)\frac{1}{6}\pi$. 32. 121 *ft.* 35. 7958 *miles.*

8. 35 *sq. yds.* 39. 86.4 *yds*, 115.2 *yds*, 64.8 *yds.*

0. $A = 71^\circ\, 15' = 79^g\, 16^{\backprime}\, 16^{\backprime\backprime}$; $B = 3^\circ\, 45' = 4^g\, 16^{\backprime}\, 66^{\backprime\backprime}$.

1. $\frac{2}{50}\pi$. 42. .8, $\frac{3}{4}$, .894. 45. $121\frac{1}{2}$ *sq. yds.*

7. 47° 44′ 47″. 49. 21 *sq. ft.* 50. 78A 1R 13P.

6. $\frac{1}{2}a \tan a$. 57. 15A 0R $33\frac{1}{5}$P. 61. $\frac{5}{8}\pi$.

8. $\dfrac{2\sqrt{(ab)}}{a+b}$, $\dfrac{2b}{a+b}$. 64. 1.8573326, 1.6532126,
3.2552726, $\bar{4}$.3802113.

8. $\dfrac{a \sin a \sin \beta}{\sin a - \sin \beta}$. 72. .01332556. 77. 45°. 78. 10 and 9.

9. $a^{\frac{1}{3}}b^{\frac{1}{3}}\sqrt{(a^{\frac{2}{3}}+b^{\frac{2}{3}})}$; $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$. 80. .016297; 56′ $1\frac{1}{2}$″.

1. 171° 53′ 14″. 82. 8A 1R 24P. 85. 25 : 4 : 9.

6. $\frac{1}{20}(10m+9n)$ *degrees*; $\frac{1}{18}(10m-9n)$ *grades.* 88. 75 *ft.*

9. $\dfrac{m-n}{m+n}$. 90. 45° or $(4n+1)\frac{1}{4}\pi$. 92. 31 *sq. in.*

3. 2R 8P 21 *sq. yds.* 94. 90°. 96. $32\frac{1}{2}$ *ft.*

7. 15 *miles.* 99. 96°, 216° : 24°, 36°, 120°. 100. 2640.

1. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2^{\frac{4}{3}}$. 103. $\pm\cos\frac{1}{2}A$, $\pm\sin\frac{1}{2}A$.

8. 59° 37′ 42″.18; 9.9358987.

0. $(-1)^m \cos\theta$, $(-1)^{m+1}\sin\theta$, $-\cot\theta$.

1. 60° or $\{3n+(-1)^n\}\frac{1}{3}\pi$. 113. $3\frac{8}{14}$°; $173\frac{4}{7}$°.

2. 30° or $(3n+1)\frac{1}{6}\pi$.

4. 90° or $(4n \pm 1)\frac{1}{2}\pi$; $\dfrac{60^\circ}{n-1}$ or $\dfrac{6n \pm 1}{3(n-1)}\pi$.

6. $\bar{1}$.8494850, $\bar{1}$.8409595, $\bar{1}$.8316166.

8. $\dfrac{2a+b}{\sqrt{3}}$. 130. .6989700, $\bar{2}$.2041200, $\bar{1}$.6989700,
.7958800, .2041200, 1.1938200.

1. $\pm(\sqrt{3}+1)$. 134. 6. 135. $a^2 - 2ac\cos 2\phi + c^2 = b^2$.

136. 295 *ft.* 137. $(n+\frac{1}{2})\pi - \frac{1}{2}a$. 138. $\sqrt{2}-1$.

139. The limits are $\pm\sqrt{2}$. 140. $\frac{1}{2}ac\sec^2\theta$, where $\tan^2\theta = \dfrac{b}{a}$.

141. $\frac{1}{2}(\sqrt{5}-1)$. 142. 40°, 80°, 140°, 100°.

144. 15° or $\{6n+(-1)^n\}\frac{1}{12}\pi$. 147. $\dfrac{a\sin a\sin\beta}{\sin(a-\beta)}$.

148. $(4n\pm 1)\frac{1}{2}\pi$. 150. $m^{\frac{2}{3}}n^{\frac{2}{3}}(m^{\frac{2}{3}}+n^{\frac{2}{3}}) = 1$.

153. (i) $\frac{2}{3}n\pi$ or $2n\pi$; (ii) $n\pi$ or $(6n\pm 1)\frac{1}{6}\pi$. 155. 1657 *ft.*

157. $n\pi$ or $n\pi + a$. 159. 3091.230. 161. $339\frac{2}{7}$ *sq. in.*

162. $\pm 30°$ or $(6n\pm 1)\frac{1}{6}\pi$. 164. 44 *ft.*

165. $\pm 45°$ or $(4n\pm 1)\frac{1}{4}\pi$. 168. 55° 46′ 16″. 169. $\sqrt{(4+2\sqrt{2})}$.

171. $k=\sqrt{(a^2+2ab\cos a+b^2)}$, $\beta=\tan^{-1}\dfrac{b\sin a}{a+b\cos a}$. 172. $\frac{1}{2}$ *mile.*

175. $A = 69°\ 10'\ 10''$, $B = 46°\ 37'\ 50''$.

176. $\frac{1}{8}\sin\theta + \frac{1}{16}\sin 3\theta - \frac{1}{16}\sin 5\theta$. 177. 30 *ft.*

178. $3\,\mathrm{L}\sec A - 4\,\mathrm{L}\cos B + 2\,\mathrm{L}\sin C = 4.90309$.

180. 300 *yds;* 4200 *sq. yds.* 182. $\cos\phi + \cos 2a = c$.

183. ± 1, ± 2. 184. $-\dfrac{1\pm\sqrt{(1+\tan^2 2A)}}{\tan 2A}$. 185. 1.06.

190. 9.4660179, 17° 8′. 192. $\pm m\sqrt{\dfrac{n^2+1}{m^2+1}}$.

194. 3 *yds.* 196. 21° 14′ 13″; 10.0305608.

197. $\dfrac{m+n}{m-n}$ or $\dfrac{m-n}{m+n}$; $\pm\dfrac{2mn}{m^2+n^2}$.

199. 45°, or $\{2n+(-1)^n\}\frac{1}{4}\pi$. 200. 126° 22′ or 96° 27′.

201. 45° or 120°, or, generally, $(4n\pm 1)\frac{1}{4}\pi$ or $(6n\pm 2)\frac{1}{3}\pi$.

204. 208 *ft.* 205. $(4n+1)\frac{1}{6}\pi$ or $(4n-1)\frac{1}{2}\pi$.

208. $a\dfrac{\sin\beta}{\sin(a+\beta)}\tan\theta$, $\phi - a$. 209. 2 *m* 629 *yds.*

210. 36° 7′ 47″. 213. $(2n+1)\frac{1}{3}\pi$ or $\dfrac{6n\pm 1}{3(n-1)}\pi$.

215. $\sqrt{(5\cdot 2\sqrt{5})}$; $\sqrt{\dfrac{2\sqrt{2}-\sqrt{3}+1}{2\sqrt{2}+\sqrt{3}-1}}$.

216. $u = k\cos(\phi\mp\theta)$, where $\tan\phi = \dfrac{b}{a}$, $k = a\sec\phi$.

217. 55° 46′ 16″. 221. $k\sin(C+\phi) = \cot a\sin b$, where

222. 0, ∞, $\sqrt{2}$. $\tan\phi = \cos b\tan A$, $k = \cos b\,\mathrm{cosec}\,\phi$.

224. $n\pi + (-1)^n\frac{1}{10}\pi$ or $n\pi - (-1)^n\frac{3}{10}\pi$. 225. 12492.

232. $\frac{285}{336}$. 233. $a\dfrac{\sin\phi}{\sin(\theta+\phi)}\left\{\dfrac{\sin\theta\sin a}{\sin(\theta+\psi)}+\sin\beta\right\}$.

6. $(2n\frac{\pi}{a}-1)\pm\frac{1}{2}(\frac{\pi}{a}-1)$. 239. $a=s\sin\frac{1}{2}A\sec\frac{1}{2}B\sec\frac{1}{2}C$.

1. $m\frac{\cos\{\frac{1}{2}(\alpha+\beta)-\gamma\}}{\sin\gamma\cos\frac{1}{2}(\alpha-\beta)}$. 243. $n\pi$ or $(6n\pm1)\frac{1}{3}\pi$.

6. $\sqrt{(\alpha^2+\alpha\beta+\beta^2)}$, $\frac{1}{4}\sqrt{3}(\alpha\beta+\alpha\gamma+\beta\gamma)$.

8. (i) $(4n\pm1)\frac{1}{4}\pi$; (ii) $(6n\pm1)\frac{1}{3}\pi$ or $(6n\pm2)\frac{1}{3}\pi$; (iii) $(8n+1)\frac{1}{4}\pi$.

1. $15°$, $\{6n+(-1)^n\}\frac{1}{12}\pi$. 252. $\tan^{-1}\frac{n}{n+2}$.

3. $\tan A=\sqrt{\frac{4n^2-m^2}{4m^2-n^2}}=\cot B$. 256. $C=90°$.

0. $[6n\pm\frac{1}{2}\{3-(-1)^m\}]\frac{1}{3}\pi$. 261. $\frac{2}{3}\sqrt{\{6(p^2+q^2)-3r^2\}}$.

2. $A=n\pi$ or $n\pi+(-1)^n\frac{1}{4}\pi$; $\text{chd}\,A=0$, or ±2, or $\pm\sqrt{(2\pm\sqrt{2})}$.

3. $u=2\sqrt{a}\cos(45°-\frac{1}{2}\theta)$, where $\cos\theta=\frac{b}{a}$.

5. $\frac{abc\tan\alpha}{4\sqrt{\{s(s-a)(s-b)(s-c)\}}}$, where $s=\frac{1}{2}(a+b+c)$.

2. $\frac{1}{12}\pi$. 274. $30°$ or $150°$; $(12n\pm1)\frac{1}{6}\pi$ or $(12n\pm5)\frac{1}{6}\pi$.

6. 8 *ft.* 277. 1.062. 278. $\sqrt{3}+2:2$.

9. $\frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$. 283. $\sqrt[4]{\frac{q^3}{pq-p^2}}$, $\sqrt[4]{\frac{p^3}{pq-p^2}}$.

4. $\frac{a\sin\alpha\sin\beta}{\sqrt{(\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos\theta)}}=\frac{a\sin\alpha\sin\beta\cos\phi}{2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)}$,

if $\tan^2\phi=\frac{\sin\alpha\sin\beta\sin^2\frac{1}{2}\theta}{\sin^2\frac{1}{2}(\alpha-\beta)\cos^2\frac{1}{2}(\alpha+\beta)}$.

5. $\frac{\sqrt{(abcd)}}{a+c}$ or $\frac{\sqrt{(abcd)}}{b+d}$. 287. 2.4065404.

8. $(4n-1)\frac{1}{4}\pi$. 291. $(4n-1)\frac{1}{4}\pi$ or $(4n-1)\frac{1}{8}\pi$.

6. (i) $a>b$, $u=\frac{\sec^2\frac{1}{2}x\cos^2\phi\sec^2\frac{1}{2}\theta}{2a}$,

where $\cos\theta=\frac{b}{a}$, $\tan\phi=\tan\frac{1}{2}\theta\tan\frac{1}{2}x$:

(ii) $a<b$, $u=\frac{\sec\frac{1}{2}(x+\theta)\sec\frac{1}{2}(x-\theta)}{2b}$, where $\cos\theta=\frac{a}{b}$.

9. $x=0$, ±1, $\pm\sqrt{7}$. 300. $a\frac{\sin\alpha\sin\varepsilon}{\sin(\delta+\varepsilon)}$, $a\frac{\sin\beta\sin\varepsilon}{\sin(\delta+\varepsilon-\gamma)}$.

3. $\pm18°$, $\pm54°$, or, in one general formula, $(n\pm\frac{1}{5}\pm\frac{1}{10})\pi$.

0. $15°$, $30°$, $45°$, $75°$, $90°$, $105°$, $135°$, $150°$, $165°$, $195°$, $210°$, $215°$, $245°$, $270°$, $275°$, $305°$, $330°$, $335°$, $365°$, $390°$:

$\theta=\frac{1}{2}n\pi$ or $(1\pm\frac{1}{3})(2n+1)\frac{1}{8}\pi$, n being any integer.

3. $152\frac{1}{2}$ *miles.* 314. $\sqrt{(AB)}$, $\frac{2B\sqrt{A}}{\sqrt{A}+\sqrt{B}}$.

317. $\frac{1}{a} + \frac{1}{b} = \frac{1}{m} + \frac{1}{n}$. 319. $B = 4^\circ\, 55'\, 11''$, $C = 168^\circ\, 27'\, 25''$.

320. 30°, 45°, 60°, or $(n+\frac{1}{6})\pi$, $(n+\frac{1}{4})\pi$, $(n+\frac{1}{3})\pi$.

322. 5 *m* 1156 *yds.* 324. $\pm\sqrt{(ab)}$.

329. $\theta = (m+n)\frac{1}{2}\pi$, $\phi = (n-m)\frac{1}{2}\pi$; 330. $x = \pm 1$ or $\frac{1}{2}$.
or $\theta = \frac{1}{2}(m+n)\pi + \{2+(-1)^m\}\frac{1}{8}\pi$, $\phi = \frac{1}{2}(m-n)\pi + \{2-(-1)^m\}\frac{1}{8}\pi$.

331. $\frac{m \sin A}{n - m\cos A} = \frac{\sin A \sin\phi}{\sin(A-\phi)}$, if $\tan\phi = \frac{m}{n}\sin A$.

333. 6.34 *miles.* 338. $h' = \sqrt{\frac{(b-c)\,(2bc-ab-ac)}{a}}$, $h = \frac{a}{b}h'$.

340. $\frac{1}{6}(4\pi - 3\sqrt{3})a^2$. 341. 4 *sq. in.*, $1\frac{3}{11}$ *sq. in.*

343. 15° or $(12n\pm 1)\frac{1}{12}\pi$. 346. $x = \frac{1}{4}(3\pm\sqrt{17})$, $y = -\frac{1}{4}(3\pm\sqrt{17})$.

348. 1 *m* 1527 *yds.* 350. ± 1, $\pm 2\sin^2\frac{1}{2}\alpha$, $\pm 2\cos^2\frac{1}{2}\alpha$.

354. $\pm(\sqrt{3}+1)$. 357. $\tan\theta = \pm 2$ or ± 1, $\tan\phi = \pm 1$ or $\pm\frac{1}{2}$.

360. $r^2(3+\sqrt{6})$.

361. $\sin(\theta+\phi) = \frac{\sin(\alpha+\beta)}{\sqrt{2}\cos\alpha\cos\beta}$, where $\cos 2\phi = \tan^2\alpha\tan^2\beta$.

366. $(a^2-cd)^2 = b^2\{(a+c)^2+(a+d)^2\}$. 370. 9.4 *m*, 15.2 *m.*

372. $0, \infty, \pm 1, \pm(\sqrt{2}\pm 1)$. 373. $\frac{m^2}{a^2} + \frac{1}{a^4}$. 374. $\frac{2ab\sin\alpha}{a^2-b^2}$.

378. $+\tan\frac{1}{4}A$, $-\cot\frac{1}{4}A$, $\pm\tan\frac{1}{4}(\pi\mp A)$. 379. $\alpha = \pm\frac{1}{4}\pi$, $\alpha' = -\frac{1}{6}\pi$ or $\sin^{-1}(\frac{1}{6})$.

381. $\sqrt{\frac{ab\,(a+b)}{a\cot^2\gamma + b\cot^2\alpha - (a+b)\cot^2\beta}}$. 382. $a\{1\pm\sqrt{\frac{2b}{1+b}}\}$.

386. $\frac{(av-bu)\sin\theta}{\sqrt{(u^2+v^2-2uv\cos\theta)}}$. 388. $\{6n+(-1)^n\}\frac{1}{12}\pi$. 389. $C = 90^\circ$.

397. $\sqrt{\frac{2}{a}}$. 398. $a\tan\alpha\sqrt{(2-\sqrt{2})}$. 399. $\tan\theta \operatorname{cosec}\frac{\theta}{2^n}$.

402. $(2n+1)\pi - \alpha$ or $\sin^{-1}\{-\sin\alpha\cos\alpha\} - \alpha$. 404. 232 *ft.*

406. $\frac{43}{700}$ *sq. yds.* 412. $\theta = (2n+1)\frac{1}{2}\pi$. 423. $\tan^{-1}na$. 424. $2^{\frac{1}{2}n}\cos\frac{1}{4}n\pi$.

425. $\frac{1}{4}\{2n+1-\sin(2n+1)\theta\operatorname{cosec}\theta\}$. 427. $2\cos 40^\circ$, $-2\cos 20^\circ$, $2\sin 10^\circ$.

432. -1, $\frac{1}{2}(1\pm\sqrt{3})$. 435. $\frac{1}{2}$, $\frac{1}{2}(4\pm\sqrt{3})$. 437. $\frac{1}{2}\operatorname{cosec}^2\theta - 2^{n-1}\operatorname{cosec}^2 2^n\theta$.

439. $\frac{3\sin\frac{1}{2}(n+1)\,\theta\sin\frac{1}{2}n\theta}{4\sin\frac{1}{2}\theta} - \frac{\sin\frac{3}{2}(n+1)\,\theta\sin\frac{3}{2}n\theta}{4\sin\frac{3}{2}\theta}$.

441. $\frac{\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}\{\cos\frac{1}{2}(n+1)\alpha + \sqrt{-1}\sin\frac{1}{2}(n+1)\alpha\}$. 442. $\frac{\sin 2nA}{2\sin A}$.

444. $\frac{1}{2}(xe^{x\cos\theta} + x^{-1}e^{x^{-1}\cos\theta})$. 448. $\frac{\sin(n+1)\theta\sin n\theta}{\sin\theta\cos\theta\cos(2n+1)\theta}$.

451. $4\log_e\frac{\sqrt{a}+\sqrt{b}}{2\sqrt[4]{a}} + 4(m\cos^2\theta - \frac{1}{2}m^2\cos^2 2\theta + \&c)$, where $m = \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

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SOLUTIONS OF EXAMPLES IN PLANE TRIGONOMETRY

PART I

BY THE REV. J. W. COLENSO, M.A.
RECTOR OF FORNCETT ST. MARY, NORFOLK,
LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

LONGMAN AND CO., LONDON.

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CAMBRIDGE:
PRINTED BY METCALFE AND PALMER.

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SOLUTIONS,

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*** In the following Solutions, the letter *u* will frequently be used, for shortness, to represent the quantity that may be given in the Text in any problem.

Ex. 1.

1. $90° - 15° \, 37' \, 2'' = 74° \, 22' \, 58''$; $90° - 142° \, 15' \, 29'' = -52° \, 15' \, 29''$; $90° - 1° \, 1' \, 3''.2 = 88° \, 58' \, 56''.8$.
2. $180° - 145° \, 15' \, 20'' = 34° \, 44' \, 40''$; $180° - (-27° \, 17' \, 7'') = 207° \, 17' \, 7''$.
3. $A + B = 90°$; $\therefore \; B = 90° - A = 90° - 9° \, 9' \, 9'' = 80° \, 50' \, 51''$.

   To reduce these angles to degrees, we have
   $$\begin{array}{r} 60\,\underline{)\,9.00} \\ 60\,\underline{)\,9.15} \\ .1525 \end{array}$$
   so that $A = 9°.1525$, and $B = 90° - A = 80°.8475$.
4. Given $A - B = 24° \, 24'$, and $A + B = 90°$; $\therefore \; 2A = 114° \, 24'$, $2B = 65° \, 36'$, whence A and B.
5. Put A for the *vertical* angle, B for each of the *base* angles; then $A + 2B = 180°$, or $A = 180° - 2B = 180° - 119° \, 59' \, 58'' = 60° \, 0' \, 2''$.
6. Given $A = B + \frac{1}{3}B = \frac{4}{3}B$, and $A + 2B = 180°$; $\therefore \; \frac{4}{3}B + 2B = 180°$, whence $B = 54°$, $A = \frac{4}{3}B = 72°$.
7. Given $\frac{1}{2}(A+B) = 60°$, and $\frac{1}{2}(A-B) = 10°$; $\therefore$, adding, $A = 70°$, and, subtracting, $B = 50°$, and $C = 180° - (A + B) = 60°$.
8. Suppose BC produced to D; then exterior $\angle ACD = 180° - C$ = (by question) $\frac{3}{2}A$, and its supp. C (by question) $= \frac{3}{2}B$; adding these equations, $180° = \frac{3}{2}(A + B)$;
   $\therefore \; A + B = \frac{2}{3} \times 180° = 120°$, and $C = 180° - (A + B) = 60°$;
   $\therefore \; B = \frac{2}{3}C = 40°$, and $A = 180° - (B + C) = 80°$.
9. Given $C - B = B - A$, where $C = 90°$; then $A + C = 2B$ = also $180° - B$; $\therefore \; 3B = 180°$, or $B = 60°$, and $A = 90° - B = 30°$.
10. Given $180° - A = 2(90° - B)$ (1), and $180° - A = 3(90° - C)$ (2)
    from (1) $B = \frac{1}{2}A$, from (2) $C = \frac{1}{3}(90° + A)$;
    $\therefore \; A + B + C = A + \frac{1}{2}A + \frac{1}{3}(90° + A) = 180°$, whence $A = 81\frac{9}{11}°$.

Ex. 2.

1. $100^g - (-15^g) = 115^g$; $100^g - 27^g.25 = 72^g.75 = 72^g\ 75^{\backprime}$;
$100^g - 32^{\backprime}\ 1^{\backprime\backprime} = 100^g - 0^g.3201 = 99^g.6799 = 99^g\ 67^{\backprime}\ 99^{\backprime\backprime}$.

2. $200^g - 1^g.0203 = 199^g.9797 = 198^g\ 97^{\backprime}\ 97^{\backprime\backprime}$;
$200^g - 89^g.0004 = 110^g.9996 = 110^g\ 99^{\backprime}\ 96^{\backprime\backprime}$;
$200^g - 225^g.1001 = -25^g.1001 = -25^g\ 10^{\backprime}\ 1^{\backprime\backprime}$.

3. Here $37^g\ 15^{\backprime}\ 7^{\backprime\backprime} = 37.15070$ *grades*,

$$\begin{array}{rl} \text{subtract } \frac{1}{10}\text{th} = & \underline{\ \ 3.71507} \\ & 33.43563 \ \ \textit{degrees} = 33^\circ\ 26'\ 8''; \\ & \underline{\qquad\quad 60} \\ & 26.13780 \\ & \underline{\qquad\quad 60} \\ & \ \ 8.2680 \end{array}$$

Again $90^\circ - 33^\circ\ 26'\ 8'' = 56^\circ\ 33'\ 52''$;
also $56^\circ\ 33'\ 52'' = 56.56444$ *degrees*, as found below,

$$\begin{array}{rl} \text{add } \frac{1}{9}\text{th} = & \underline{\ \ 6.28493} \\ & 62.84937 \ \ \textit{grades} = 62^g\ 84^{\backprime}\ 93^{\backprime\backprime} \end{array}$$

$$\begin{array}{r} 60\overline{)\,52.00000} \\ 60\overline{)\,33.86666} \\ \overline{56.56444} \end{array}$$

4. $1^\circ = \frac{10}{9}^g = 1^g.1111 = 1^g\ 11^{\backprime}\ 11^{\backprime\backprime}$; $1' = \frac{1}{60}$ of $1^\circ = \frac{1}{60}$ of $1^g.1111$ $= 0^g.018518$, and $1'' = \frac{1}{60}$ of $1' = \frac{1}{60}$ of $0^g.018518 = .000308 = 3^{\backprime\backprime}.08$;
$1^g = \frac{9}{10}^\circ = \frac{9}{10}$ of $60' = 54'$; $1^{\backprime} = \frac{1}{100}$ of $1^g = \frac{1}{100}$ of $54' = 0'.54 = 32''.4$.

5. If $n = 5$, then $\dfrac{n-2}{n}180^\circ = \frac{3}{5} \times 180^\circ = 108^\circ$;
if $n = 6$, then $\dfrac{n-2}{n}180^\circ = \frac{4}{6} \times 180^\circ = 120^\circ$.

6. Here $\frac{3}{5} \times 180^\circ : \frac{10}{12} \times 180^\circ :: \frac{3}{5} : \frac{10}{12} :: 72 : 80 :: 9 : 10$.

7. The vertical angle (17) $= \frac{1}{7} \times 180^\circ$; hence the two base-angles together $= 180^\circ - \frac{1}{7} \times 180^\circ = \frac{6}{7} \times 180^\circ$, and each of them $= \frac{3}{7} \times 180^\circ$; hence the ratio required is $\frac{1}{7} : \frac{3}{7}$, or $1 : 3$.

8. Let the angles be $2A$, $3A$, $4A$, $5A$, $6A$; then their sum $= 20A = (n-2)\ 180^\circ$ by (16), where $n = 5$;
that is, $20A = 3 \times 180^\circ = 540^\circ$, and $A = 27^\circ$.

9. In the figure, *Euc.* IV. 10, since each of the angles at the base of the triangle ABD is double of the vertical angle, we have $\angle(ABD + ADB + BAD) = 5\angle BAD = 180^\circ$; hence $\angle BAD = 36^\circ = \frac{360^\circ}{10}$, so that (17) BD, (or its equal AC,) is *the side of a* decagon inscribed in the larger circle.

Also $\angle CDA = \angle CAD = 36^\circ = \frac{180^\circ}{5} =$ (17) angle subtended at the circumference of a circle by a side of its inscribed pentagon; $\therefore AC =$ side of pentagon inscribed in smaller circle.

. Let the polygons be of m and n sides respectively: then (16) sum of angles in the first $= (m-2)\,180^\circ$, and in the second $= (n-2)\,180^\circ$: hence (1) $(m-2)\,180^\circ - (n-2)\,180^\circ = 360^\circ$, whence $m - n = 2$, or $m = n + 2$;

and (2) $(m-2)\,180^\circ : (n-2)\,180^\circ :: 5 : 3 :: m-2 : n-2$, whence $5n - 10 = 3m - 6 = 3n$ by (1), so that $n = 5$, $m = 7$.

. By (16), an angle of the first $= \frac{2n-2}{2n}\,180^\circ = \frac{n-1}{n}\,180^\circ$, and of the second $= \frac{n-2}{n}\,180^\circ$; hence $\frac{n-1}{n}\,180^\circ = 1\frac{1}{3} \times \frac{n-2}{n}\,180^\circ$, or $3(n-1) = 4(n-2)$, whence $n = 5$, and the angles are $\frac{4}{5} \times 180^\circ = 144^\circ$ and $\frac{3}{5} \times 180^\circ = 108^\circ$.

. Since, in A. P., $S = \{2a + (n-1)d\}\,\frac{1}{2}n$, we have here by (16) $(n-2)\,180^\circ = S = \{240^\circ + (n-1)\,5^\circ\}\,\frac{1}{2}n$, whence

$$n^2 - 25n + 144 = 0, \text{ and } n = 16 \text{ or } 9.$$

Ex. 3.

. $a = \frac{1}{180}$ of semicircumference $= \frac{1}{180}\pi r = \frac{1}{180} \times \frac{22}{7} \times 9 = .157$ *in.*

. $\theta = \frac{arc}{rad} = \frac{30}{25} = \frac{6}{5}$; $\therefore A = \theta\omega^\circ = \frac{6}{5} \times 57^\circ.29577 = 68^\circ\,45'\,18''$.

. $\theta = \frac{arc}{rad} = \frac{40\,ft}{2\frac{1}{2}\,yds} = \frac{40}{2\frac{1}{2} \times 3} = 5\frac{1}{3}$: $\theta = \frac{A}{180}\pi = \frac{40}{180}\pi = \frac{2}{9}\pi$.

. The circumference of tree $= 18\,ft = 2\pi r$; $\therefore$ thickness

$$= 2r = \frac{18}{\pi} = \frac{1}{10} \times \frac{180}{\pi} = \frac{1}{10} \times 57.29577 = 5.729577\,ft = 5\,ft\ 9\,in.$$

. Area $= \pi r^2 = \frac{81}{\pi} = 81 \times \frac{7}{22} = 25.77$ *sq. ft:* hence quantity of wood in 30 feet of such timber $= 30 \times 25.77 = 773$ *cub. ft.*

. Let $ABCD$ be the square, O the centre of circle, whose radius $r = \frac{1}{2}\,yd = \frac{3}{2}\,ft$; then area of square $= AB^2 = AO^2 + BO^2 = 2r^2$, and area remaining

$$= \pi r^2 - 2r^2 = (\pi - 2)\,r^2 = 1.14159 \times \frac{9}{4} = 2.57\ sq.\,ft.$$

. An acre $= 4840$ *sq. yds* $= \pi r^2$;

$$\therefore r^2 = \frac{4840}{\pi} = 4840 \times \frac{7}{22} = 1540 \text{ sq. yds};$$

and $r = \sqrt{1540} = 39.24$ *yds.*

8. Here 60° of circle, radius r, $= \frac{1}{3}\pi r$, and 45° of circle, radi $= \frac{1}{4}\pi r'$; $\therefore \frac{1}{3}\pi r = \frac{1}{4}\pi r'$, or $r : r' :: 3 : 4$, and the areas a $r^2 : r'^2 = 9 : 16$.

9. $2\pi r = 2\pi \times \frac{3}{5} = \frac{6}{5}\pi$ *inches*, $2\pi r' = 2\pi \times \frac{3}{4} = \frac{3}{2}\pi$ *in.*, $2\pi r'' = 2\pi$ $= \frac{3}{5}\pi$ *in.*; and while the hour-hand makes *one* revol (that is, in 12 hours,) the minute-hand makes 12, an second-hand 12×60 revolutions; hence the rates of m of their extremities are as $\frac{6}{5}\pi : 12 \times \frac{3}{2}\pi : 12 \times 60 \times \frac{3}{5}\pi$, $1 : 15 : 360$.

10. The circumference of her orbit $= 2\pi r = 2 \times \frac{22}{7} \times 96{,}00$ $= \frac{4{,}224{,}000{,}000}{7}$ miles; and her rate *per* minute

$$= \frac{4224000000}{7 \times 365 \times 24 \times 60} = 1148 \text{ miles.}$$

11. a = length of $1° = \frac{1}{180}\pi r = \frac{1}{180} \times \frac{22}{7} \times 4000 = 69.8$ miles; length round equator $= 2\pi r = 2 \times \frac{22}{7} \times 4000 = 25{,}143$ *mi*

12. The Earth turns round on its axis once in every 24 h hence each point in the Equator will in 24 hours des a circle whose radius is 4000 miles; ∴ rate of this m *per* minute $= \frac{2\pi \times 4000}{24 \times 60} = \frac{1100}{63} = 17\frac{1}{2}$ *miles.*

Ex. 4.

1. $\theta = \frac{A}{180}\pi = \frac{72}{180}\pi = \frac{2}{5}\pi$; $\theta' = \frac{\pi}{2} - \theta = \frac{1}{10}\pi$; $\theta'' = \pi - \theta =$

2. $6° \, 3' \, 36'' = 6°.06$; and $\theta = \frac{6.06}{180}\pi = \frac{606}{18000}\pi = \frac{101}{3000}\pi$.

3. Let the angles be 3θ, 4θ, 5θ; then their sum $= 12\theta =$ (13): hence $\theta = \frac{1}{12}\pi$, &c.

4. $\theta = \frac{arc}{rad} = \frac{\pi \, feet}{\pi \, yards} = \frac{\pi}{3\pi} = \frac{1}{3}$;

$$A = \theta\omega° = \tfrac{1}{3} \times 57°.29577 = 19°.09859 = 19° \, 5' \, 55''.$$

5. $\theta = .1 = \frac{1}{10}$; $\therefore A = \theta\omega° = \frac{1}{10} \times 57°.29577 = 5°.729577 = 5° \, 43$

6. $\theta = \frac{A}{200}\pi$ by (30) $= \frac{2.1875}{200}\pi = \frac{21875}{2000000}\pi = \frac{7}{640}\pi$.

7. $\theta : \theta' :: \frac{a}{r} : \frac{a'}{r'} :: a^2 : a'^2 :: \frac{4}{9} : \frac{9}{25} :: 100 : 81$.

8. $90° - \frac{1}{2}\pi° = 90° - \frac{11}{7}° = 88\frac{3}{7}°$; $\theta = \frac{A}{180}\pi = \frac{180 - 2\pi}{180}\pi$, (

A = *supplement of* $2\pi° = 180° - 2\pi°$,) $= \frac{180 - \frac{44}{7}}{180}\pi = \frac{30}{3}$

9. The length of the arc $a = \theta r = 3\frac{3}{4} \times 2\frac{2}{3} = 10$; $\theta' = \frac{arc}{rad} = \frac{10}{2\frac{1}{2}} = 4$:

$$A = \theta\omega^\circ = 3\tfrac{3}{4} \times 57^\circ.29577 = 214^\circ.85913 = 214^\circ\ 51'\ 33'';$$
$$A' = \theta'\omega^\circ = 4 \times 57^\circ.29577 = 229^\circ.18308 = 229^\circ\ 10'\ 59''.$$

10. Here $\frac{1}{4}\pi r = \frac{1}{3}\pi r'$, or $r' = \frac{3}{4}r$; and $\theta = \frac{arc}{rad} = \frac{r'}{r} = \frac{3}{4}$.

11. Let θ be the circular measure of the angle in question: then $\frac{a}{r} = \frac{\pi}{2} - \theta$, and, if $\frac{a'}{r'} = \pi - \theta$, we have $\frac{a'}{r'} = \frac{\pi}{2} + \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2} + \frac{a}{r}$.

12. Here $2r = \pi r'$, and $\theta = \frac{arc}{rad} = \frac{2r'}{r} = \frac{4}{\pi}$;

$$\therefore A = \theta\omega^\circ = \frac{4}{\pi} \times \frac{180^\circ}{\pi} = \frac{8820^\circ}{121}, \text{ (putting } \pi = \frac{22}{7}\text{,)} = 72^\circ 53' 33''.$$

Ex. 5.

1. $\sin = \sqrt{(1 - \cos^2)} = \sqrt{(1 - \frac{1}{\sec^2})} = \frac{\sqrt{(\sec^2 - 1)}}{\sec}$; $\sin = \frac{1}{\operatorname{cosec}} = \frac{1}{\sqrt{(1 + \cot^2)}}$.

2. $\cos = \sqrt{(1 - \sin^2)} = \sqrt{(1 - \frac{1}{\operatorname{cosec}^2})} = \frac{\sqrt{(\operatorname{cosec}^2 - 1)}}{\operatorname{cosec}}$; and, since $\cot = \frac{\cot}{1} = \frac{\cos}{\sin}$, $\therefore$ (40) $\cos = \frac{\cot}{\sqrt{(1 + \cot^2)}}$.

3. $\tan = \frac{\sin}{\cos} = \frac{\sin}{\sqrt{(1 - \sin^2)}}$; $\tan = \frac{1}{\cot} = \frac{1}{\sqrt{(\operatorname{cosec}^2 - 1)}}$.

4. $\cot = \frac{\cos}{\sin} = \frac{\cos}{\sqrt{(1 - \cos^2)}}$; $\cot = \frac{1}{\tan} = \frac{1}{\sqrt{(\sec^2 - 1)}}$.

5. $\sec = \sqrt{(1 + \tan^2)} = \sqrt{(1 + \frac{1}{\cot^2})} = \frac{\sqrt{(1 + \cot^2)}}{\cot}$;

$$\operatorname{cosec} = \frac{1}{\sin} = \frac{1}{\sqrt{(1 - \cos^2)}} = \frac{1}{\sqrt{(1 - \frac{1}{\sec^2})}} = \frac{\sec}{\sqrt{(\sec^2 - 1)}}.$$

6. $\sin = \sqrt{(1 - \cos^2)} = \sqrt{\{1 - (1 - \text{vers})^2\}} = \sqrt{(2\,\text{vers} - \text{vers}^2)}$.

7. $\sin = \frac{3}{\sqrt{(9 + 16)}} = \frac{3}{5}$, $\cos = \frac{4}{5}$, $\cot = \frac{4}{3}$, $\sec = \frac{5}{4}$, $\operatorname{cosec} = \frac{5}{3}$.

8. $\cos = \sqrt{(1 - \sin^2)} = \sqrt{(1 - \frac{4}{9})} = \frac{1}{3}\sqrt{5}$;

$$\therefore \cot = \frac{\cos}{\sin} = \frac{\sqrt{5}}{2}, \quad \sec = \frac{1}{\cos} = \frac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5}.$$

9. $\sin = \frac{1}{\sqrt{(1 + 4)}} = \frac{1}{\sqrt{5}}$, and $\therefore \operatorname{cosec} = \sqrt{5}$: $\cos = \frac{2}{\sqrt{(1 + 4)}} = \frac{2}{\sqrt{5}}$

10. $\cos = \frac{1}{\sec} = \frac{1}{4}$; $\therefore \sin = \sqrt{(1-\cos^2)} = \sqrt{(1-\frac{1}{16})} = \frac{\sqrt{15}}{4}$, and $\text{vers} = 1 - \cos = \frac{3}{4}$.

11. $(\sin\frac{1}{6}\pi + \cos\frac{1}{6}\pi)(\sin\frac{1}{3}\pi - \cos\frac{1}{3}\pi) = (\frac{1}{2} + \frac{1}{2}\sqrt{3})(\frac{1}{2}\sqrt{3} - \frac{1}{2})$

$$= \tfrac{3}{4} - \tfrac{1}{4} = \tfrac{1}{2} = \sin\tfrac{1}{6}\pi.$$

12. $\frac{1-\tan^2 30^\circ}{1+\tan^2 30^\circ} = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{1}{2} = \cos 60^\circ$:

$$\frac{\sin 45^\circ - \sin 30^\circ}{\sin 45^\circ + \sin 30^\circ} = \frac{\frac{1}{2}\sqrt{2} - \frac{1}{2}}{\frac{1}{2}\sqrt{2} + \frac{1}{2}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{\sec 45^\circ - \tan 45^\circ}{\sec 45^\circ + \tan 45^\circ};$$

$$\text{or } = \frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = (\sqrt{2}-1)^2 = (\sec 45^\circ - \tan 45^\circ)^2.$$

Ex. 6.

1. A = (in *degrees*) $\frac{1}{2} \times 57.29577 = 28.647885$, and $\therefore$ = (in *grades*) $\frac{10}{9} \times 28.647885 = 31.83098 = 31^g\ 83'\ 10''$.

2. Here $2\pi r = \frac{1}{3}\pi r'$, or $r' = 6r$; and the areas are as

$$r^2 : r'^2 :: 1 : 36.$$

3. Let their distances be $7r$ and $10r$; then their orbits are $2\pi \times 7r$ and $2\pi \times 10r$; hence their rates of motion are as $\frac{14\pi r}{224} : \frac{20\pi r}{365}$, or as $73 : 64$.

4. $\theta = \frac{3\frac{1}{3}}{2\frac{1}{2}} = \frac{4}{3}$, $\theta' = \frac{2\frac{1}{3}}{3\frac{1}{2}} = \frac{2}{3}$; $\therefore \theta : \theta' :: 2 : 1$:

also $A = \theta\omega^\circ = \frac{4}{3} \times 57^\circ.29577 = 76^\circ.39436 = 76^\circ\ 23'\ 40''$;

and A' = (in *degrees*) $\frac{1}{2}A = 38.19718$

= (in *grades*) $\frac{10}{9} \times 38.19718 = 42^g\ 44'\ 13''$.

5. $\left(\frac{1}{2}\right)^2 : \left(\frac{1}{\sqrt{2}}\right)^2 : \left(\frac{\sqrt{3}}{2}\right)^2 : 1^2 :: \frac{1}{4} : \frac{1}{2} : \frac{3}{4} : 1 :: 1 : 2 : 3 : 4.$

6. $(1-\cos^2) + 5\cos^2 = 3$, or $\cos^2 = \frac{1}{2}$, and $\sec^2 = 2 = 1 + \tan^2$;

$$\therefore \tan^2 = 1, \text{ and } \tan = \pm 1.$$

7. $6\sin - 3\cos = 4\sin + 6\cos$, or $2\sin = 9\cos$;

$$\therefore \tan = \frac{9}{2}, \text{ and } \sin = \frac{9}{\sqrt{(81+4)}} = \frac{9}{\sqrt{85}},\ \cos = \frac{2}{\sqrt{85}}.$$

8. $\tan\theta + \frac{3}{\tan\theta} = 4$, or $\tan^2\theta - 4\tan\theta + 3 = 0$;

$$\therefore \tan\theta = 1 \text{ or } 3, \text{ and } \sin\theta = \frac{1}{\sqrt{(1+1)}} = \frac{1}{\sqrt{2}}, \text{ or } = \frac{3}{\sqrt{(9+1)}} = \frac{3}{\sqrt{10}}.$$

. Since $\cos(90^\circ - x) = \sin x$, we have

$$1 = \cos^2 x + \sin^2 x = \frac{\cos^2 A}{\sin^2 C} + \frac{\cos^2 B}{\sin^2 C};$$

$$\therefore \sin^2 C = \cos^2 A + \cos^2 B = (1 - \sin^2 A) + (1 - \sin^2 B), \text{ \&c.}$$

. Here we must first find $\cos A$: now $(\sec^2 A - 1) + 4(1 - \cos^2 A) = 6$, or $\dfrac{1}{\cos^2 A} - 4\cos^2 A = 3$; $\therefore 4\cos^4 A - 3\cos^2 A - 1 = 0$, or $\cos^2 A = \frac{1}{4}$, (the other root -1 giving impossible values for $\cos A$;) hence $\cos A = \pm\frac{1}{2}$, and $\operatorname{vers} A = 1 - \cos A = \frac{1}{2}$ or $\frac{3}{2}$.

. By (40), $$\frac{a\sin + b\cos}{a\sin - b\cos} = \frac{\dfrac{a^2}{\sqrt{(a^2+b^2)}} + \dfrac{b^2}{\sqrt{(a^2+b^2)}}}{\dfrac{a^2}{\sqrt{(a^2+b^2)}} - \dfrac{b^2}{\sqrt{(a^2+b^2)}}} = \frac{a^2+b^2}{a^2-b^2};$$

$$1 - \frac{\sin^2}{1+\cot} - \frac{\cos^2}{1+\tan} = 1 - \frac{\dfrac{a^2}{a^2+b^2}}{1+\dfrac{b}{a}} - \frac{\dfrac{b^2}{a^2+b^2}}{1+\dfrac{a}{b}}$$

$$= 1 - \frac{a^3}{(a+b)(a^2+b^2)} - \frac{b^3}{(a+b)(a^2+b^2)}$$

$$= 1 - \frac{a^3+b^3}{(a+b)(a^2+b^2)} = 1 - \frac{a^2 - ab + b^2}{a^2+b^2} = \frac{2ab}{a^2+b^2}.$$

. $\dfrac{\sin A}{\tan A} = \dfrac{\sqrt{2}\sin B}{\sqrt{3}\tan B}$, or $\cos A = \sqrt{\frac{2}{3}}\cos B$; hence
$1 = \sin^2 A + \cos^2 A = 2\sin^2 B + \frac{2}{3}\cos^2 B = 2\sin^2 B + \frac{2}{3} - \frac{2}{3}\sin^2 B$:
$\therefore \sin B = \pm\frac{1}{2}$, or $B = \pm 30^\circ$, and $\sin A = \sqrt{2}\sin B = \pm\dfrac{1}{\sqrt{2}}$, or $A = \pm 45^\circ$.

Ex. 7.

1. $\sin = \dfrac{\sin}{\cos} \times \cos = \tan \times \cos = \dfrac{\cos}{\cot} = \dfrac{\cos}{\sqrt{(\operatorname{cosec}^2 - 1)}}$; or, to radius r, $\operatorname{Sin} = \dfrac{r\operatorname{Cos}}{\sqrt{(\operatorname{Cosec}^2 - r^2)}}$, which may be *proved* by writing $\dfrac{\operatorname{Sin}}{r}$, $\dfrac{\operatorname{Cos}}{r}$, $\dfrac{\operatorname{Cosec}}{r}$, for sin, cos, cosec, respectively, in the former result; and, similarly, in all the following instances.

2. $\operatorname{vers} = 1 - \cos = 1 - \dfrac{1}{\sec} = \dfrac{\sec - 1}{\sec}$; $\operatorname{vers} = (1 - \cos) \times \dfrac{\sin}{\sin}$

$$= \left(\frac{1}{\sin} - \frac{\cos}{\sin}\right)\sin = (\operatorname{cosec} - \cot)\frac{1}{\operatorname{cosec}} = \frac{\operatorname{cosec} - \cot}{\operatorname{cosec}}.$$

3. $\sin.\cos = \dfrac{1}{\dfrac{1}{\sin.\cos}} = \dfrac{1}{\dfrac{\sin^2+\cos^2}{\sin.\cos}} = \dfrac{1}{\dfrac{\sin}{\cos}\times\dfrac{\cos}{\sin}} = \dfrac{1}{\tan+\cot}$.

4. $\cot^2(1-\sin^2) = \cot^2 - \dfrac{\cos^2}{\sin^2}\times\sin^2 = \cot^2-\cos^2$.

5. $\sec^2(1+\cot^2) = \sec^2 + \dfrac{1}{\cos^2}\times\dfrac{\cos^2}{\sin^2} = \sec^2+\dfrac{1}{\sin^2} = \sec^2+\text{cosec}^2$.

6. $(\tan+\cot) = \sqrt{(\tan+\cot)^2} = \sqrt{(\tan^2+2+\cot^2)}$, since $2\tan.\cot = 2$,
$= \sqrt{\{(1+\tan^2)+(1+\cot^2)\}} = \sqrt{(\sec^2+\text{cosec}^2)}$.

Refer now to the figure in (50).

7. By similar triangles PNA, ADT', $PN : AP :: AD : AT'$, or $PN = \dfrac{AP.AD}{AT'}$, that is, $\sin = \dfrac{1\times1}{\text{cosec}} = \dfrac{1}{\text{cosec}}$.

8. By similar triangles, TBA, ADT', $BT : AB :: AD : DT'$, or $BT.DT' = AB.AD$, that is, $\tan.\cot = 1\times1 = 1$.

9. By similar triangles, ABT, ANP, $BT : AB :: PN : AN$, or $BT = \dfrac{AB.PN}{AN}$, that is, $\tan = \dfrac{1\times\sin}{\cos} = \dfrac{\sin}{\cos}$.

10. In the figure of (56), draw CN perpendicular to AB; then by similar triangles, BCN, BAD, $CN : CB :: AD : AB$, or $CN = \dfrac{CB.AD}{AB}$, that is,
$$\sin A = \frac{\text{chd}\,A.\cos\frac{1}{2}A}{1} = 2\sin\tfrac{1}{2}A\,\cos\tfrac{1}{2}A.$$

11. In the same similar triangles, $BN : BC :: BD : AB$, or $BN = \dfrac{BC.BD}{AB}$, that is, $\text{vers}\,A = 1-\cos A = \dfrac{\text{chd}\,A.\sin\frac{1}{2}A}{1}$
$= 2\sin\frac{1}{2}A.\sin\frac{1}{2}A$, or $\cos A = 1-2\sin^2\frac{1}{2}A$.

12. In the same figure, produce BA to meet the circumference again in B'; then, in similar triangles, $BB'C$, BCN, $BC : BB' :: BN : BC$, or $BC^2 = BB'.BN$, that is,
$$\text{chd}^2 A = 2\times\text{vers}\,A = 2\,\text{vers}\,A.$$

Ex. 3.

1. $\cos 150° = \cos(180°-30°) = -\cos 30° = -\frac{1}{2}\sqrt{3}$,
$\text{cosec}\,150° = \text{cosec}\,30° = 2$;
$\sin 135° = \sin 45° = \frac{1}{2}\sqrt{2}$, $\tan 135° = -\tan 45° = -1$;
$\tan 120° = -\tan 60° = -\sqrt{3}$, $\sec 120° = -\sec 60° = -2$.

2. $\sin 240^\circ = \sin(180^\circ + 60^\circ) = -\sin 60^\circ = -\frac{1}{2}\sqrt{3}$,
$\cos 240^\circ = -\cos 60^\circ = -\frac{1}{2}$, $\tan 240^\circ = +\tan 60^\circ = \sqrt{3}$.

3. $\text{chd}\,120^\circ = 2\sin 60^\circ = \sqrt{3}$, $\text{vers}\,120^\circ = 1 - \cos 120^\circ = 1 + \cos 60^\circ = 1\frac{1}{2}$;
$\text{chd}\,300^\circ = 2\sin 150^\circ = 2\sin 30^\circ = 1$,
$\text{vers}\,300^\circ = 1 - \cos 300^\circ = 1 - \cos(360^\circ - 60^\circ) = 1 - \cos 60^\circ = \frac{1}{2}$.

4. $\text{covers}\,330^\circ = 1 - \sin 330^\circ = 1 - \sin(360^\circ - 30^\circ) = 1 + \sin 30^\circ = 1\frac{1}{2}$;
$\text{suvers}\,315^\circ = 1 + \cos 315^\circ = 1 + \cos(360^\circ - 45^\circ) = 1 + \cos 45^\circ$
$= 1 + \frac{1}{2}\sqrt{2}$.

5. $\sin 100^\circ = \sin(90^\circ + 10^\circ) = +\cos 10^\circ$;
$\cos 200^\circ = \cos(180^\circ + 20^\circ) = -\cos 20^\circ$;
$\tan 300^\circ = \tan(3.90^\circ + 30^\circ) = -\cot 30^\circ$;
$\cot 400^\circ = \cot(4.90^\circ + 40^\circ) = +\cot 40^\circ$;
$\sec 500^\circ = \sec(6.90^\circ - 40^\circ) = -\sec 40^\circ$;
$\text{cosec}\,600^\circ = \text{cosec}(7.90^\circ - 30^\circ) = -\sec 30^\circ$.

6. $1 + \{\cos\theta + \sin\theta\} \times \{-\sin\theta + \cos\theta\} = 1 + \cos^2\theta - \sin^2\theta = 2\cos^2\theta$.

7. $(-\sin\theta) \times (+\cos\theta) + (-\sin\theta) \times (-\cos\theta) + (+\tan\theta) \times (+\cot\theta)$
$= -\sin\theta\cos\theta + \sin\theta\cos\theta + 1 = 1$.

8. $(+\tan A) + (-\tan A) + (-\tan A) - (+\tan A) + (+\tan A) + (+\tan A) = 0$.

9. $\sin\{2n\pi + (\frac{1}{2}\pi \pm \theta)\} = \sin(\frac{1}{2}\pi \pm \theta) = +\cos\theta$;
$\tan\{2n\pi + (\frac{3}{2}\pi \pm \theta)\} = \tan(\frac{3}{2}\pi \pm \theta) = \mp\cot\theta$.

10. $\{1 - \cos(90^\circ + A)\}\{1 - \cos(A - 90^\circ)\}$
$+ \{1 - \sin(270^\circ - A\}\{1 - \sin(A - 270^\circ)\}$
$= (1 + \sin A)(1 - \sin A) + (1 + \cos A)(1 - \cos A) = 2 - \sin^2 A - \cos^2 A = 1$.

1. Here $(-\sec\theta)(-\sin a)(-\cos a) = (+\cot a)(-\cos\theta)$,

$$\text{or } \sec\theta \sin a \cos a = \cot a \cos\theta = \frac{\cos a}{\sin a \sec\theta};$$

$$\therefore \sec^2\theta = \frac{1}{\sin^2 a} = \text{cosec}^2 a, \text{ and } \sec\theta = \pm\,\text{cosec}\,a.$$

2. $\text{suvers}(90^\circ - A) + \text{covers}(180^\circ - A) = \{1 + \cos(90^\circ - A)\}$
$+ \{1 - \sin(180^\circ - A)\} = (1 + \sin A) + (1 - \sin A) = 2$.

Ex. 9.

1. $\tan A = 1 = \tan\frac{1}{4}\pi$; $\therefore A = n\pi + \frac{1}{4}\pi = (4n + 1)\frac{1}{4}\pi$.

2. $\tan 2A = \pm 1 = \tan \pm \frac{1}{4}\pi$; $\therefore 2A = n\pi \pm \frac{1}{4}\pi = (4n \pm 1)\frac{1}{4}\pi$,
and $A = (4n \pm 1)\frac{1}{8}\pi$.

3. $\sec A = -2 = \sec\frac{2}{3}\pi$; $\therefore A = 2n\pi \pm \frac{2}{3}\pi = (6n \pm 2)\frac{1}{3}\pi$.

4. $\cos 2A = \frac{1}{2} = \cos\frac{1}{3}\pi$; $\therefore 2A = 2n\pi \pm \frac{1}{3}\pi = (6n \pm 1)\frac{1}{3}\pi$,
and $A = (6n \pm 1)\frac{1}{6}\pi$.

5. $\tan 5A = \sqrt{3} = \tan\frac{1}{3}\pi$; $\therefore\ 5A = n\pi + \frac{1}{3}\pi = (3n+1)\frac{1}{3}\pi$, and $A = (3n+1)\frac{1}{15}\pi$.

6. Since $\tan A = \tan(n\pi + a)$, we must expect the value of $\sec A$ found from that of $\tan A$, to give that of $\sec(n\pi + a)$, whic $= \pm\sec a$, according as n is even or odd.

7. Since $\sin A = \sin\{n\pi + (-1)^n a\}$, and $\cos A = \cos(2n\pi \pm a)$, $\therefore$ $\cos 2A$, in terms of $\sin A$, will give $\cos\{2n\pi + (-1)^n 2a\} = \cos 2$ and $\cos 2A$, in terms of $\cos A$, will give $\cos(4n\pi \pm 2a) = \cos 2$ so that, in either case, $\cos 2A$ will have but *one* value. Bu $\sin 2A$, in terms of $\sin A$, will give $\sin\{2n\pi + (-1)^n 2a\} = \pm\sin 2$ + or − according as n is even or odd, and $\sin 2A$, in terms $\cos A$, will give $\sin(4n\pi \pm 2a) = \pm\sin 2a$; so that, in eith case, $\sin 2A$ will have *two* values.

8. Referring to the figure in (66), $BC = AB\sin 60° = 30 \times \frac{1}{2}\sqrt{}$ $= 15 \times 1.7320\,\&c = 25.98\,ft = 26\,ft$; and $AC = AB\cos 60° = 15$

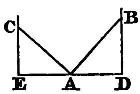
9. In the same figure, $\angle BAC = \angle$ of depression of A, as see from B, $= 30°$; and

$$AC = BC\cot 30° = 90 \times \sqrt{3} = 90 \times 1.7320\,\&c = 156\,ft.$$

10. In the same figure, length of maypole $= CB + BA = AC\tan 45$ $+ AC\sec 45° = 21(1 + \sqrt{2}) = 21 \times 2.414\,\&c = 51\,ft.$

11. Here height of tower (in *yards*) : height of rule ($= 2\,ft$) :: length of shadow of tower ($= 56$ *yds*) : length of shadow of rule ($= 4\frac{1}{4}\,ft$), or height of tower $= \dfrac{2}{4\frac{1}{4}} \times 56$ *yds* $= 79\,ft.$

12. Let AB, AC, be the two positions of the ladder, BD, CE, the heights of the two windows; then, since $\angle BAC = 90°$, $\angle CAE$ is the comp. of $\angle BAD$.



Now $\sin BAD = \dfrac{BD}{AB} = \dfrac{27}{45} = \dfrac{3}{5}$;

and $CE = AC\sin CAE = AC\cos BAD = 45 \times \frac{4}{5} = 36\,ft$;
also $DE = AD + AE = AB\cos BAD + AC\cos CAE$
$= AB(\cos BAD + \sin BAD) = 45(\frac{3}{5} + \frac{4}{5}) = 63\,ft.$

Ex. 10.

1. $\cos(60° - A) = \cos 60°\cos A + \sin 60°\sin A = \frac{1}{2}\cos A + \frac{1}{2}\sqrt{3}\sin A$;

$$\sin(45° + A) = \sin 45°\cos A + \cos 45°\sin A = \frac{1}{\sqrt{2}}\cos A + \frac{1}{\sqrt{2}}\sin A.$$

2. $\{1 - \cos(A + 30°)\} - \{1 - \cos(A - 30°)\} = \cos(A - 30°) - \cos(A + 30°)$ $= (\cos A\cos 30° + \sin A\sin 30°) - (\cos A\cos 30° - \sin A\sin 30°)$ $= 2\sin 30°\sin A = \sin A.$

3. $$\frac{\frac{1}{\sqrt{2}}(\cos A+\sin A)-\frac{1}{\sqrt{2}}(\cos A-\sin A)}{\frac{1}{\sqrt{2}}(\cos A+\sin A)+\frac{1}{\sqrt{2}}(\cos A-\sin A)}=\frac{\frac{1}{\sqrt{2}}(2\sin A)}{\frac{1}{\sqrt{2}}(2\cos A)}=\tan A.$$

4. $$\frac{\frac{1+\tan A}{1-\tan A}-\frac{1-\tan A}{1+\tan A}}{\frac{1+\tan A}{1-\tan A}+\frac{1-\tan A}{1+\tan A}}=\frac{(1+\tan A)^2-(1-\tan A)^2}{(1+\tan A)^2+(1-\tan A)^2}=\frac{4\tan A}{2+2\tan^2 A}$$

$$=\frac{2\tan A}{\sec^2 A}=\frac{2\sin A}{\cos A}\times\cos^2 A=2\sin A\cos A.$$

5. $$\sin(n+1)A+\sin(n-1)A=\sin(nA+A)+\sin(nA-A)$$
$$=(\sin nA\cos A+\cos nA\sin A)+(\sin nA\cos A-\cos nA\sin A)$$
$$=2\sin nA\cos A,\ \&c:$$
$$\cos(n+1)A+\cos(n-1)A=\cos(nA+A)+\cos(nA-A)=\&c.$$

Put $n=1$ in the two given formulæ; then

$$\sin 2A=2\sin A\cos A-\sin 0=2\sin A\cos A,$$
$$\cos 2A=2\cos A\cos A-\cos 0=2\cos^2 A-1.$$

6. $$\cot(A\pm B)=\frac{\cos(A\pm B)}{\sin(A\pm B)}=\frac{\cos A\cos B\mp\sin A\sin B}{\sin A\cos B\pm\cos A\sin B}$$
$$=\frac{\cot A\cot B\mp 1}{\cot B\pm\cot A},$$ dividing each term by $\sin A\sin B$.

7. $$\sec(A\pm B)=\frac{1}{\cos(A\pm B)}=\frac{1}{\cos A\cos B\mp\sin A\sin B}$$
$$=\frac{1}{\cos A\cos B(1\mp\tan A\tan B)}=\frac{\sec A\sec B}{1\mp\sqrt{\{(\sec^2 A-1)(\sec^2 B-1)\}}}.$$

8. $$\operatorname{cosec}(A\pm B)=\frac{1}{\sin(A\pm B)}=\frac{1}{\sin A\cos B\pm\cos A\sin B}$$
$$=\frac{1}{\sin A\sin B(\cot B\pm\cot A)}=\frac{\operatorname{cosec}A\operatorname{cosec}B}{\sqrt{(\operatorname{cosec}^2 B-1)}\pm\sqrt{(\operatorname{cosec}^2 A-1)}}.$$

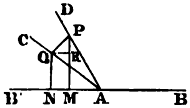
9. Given $\sin(A-B)=\sin A\cos B-\cos A\sin B$:

then $$\cos(A-B)=\sin\{90^\circ-(A-B)\}=\sin\{(90^\circ-A)+B\}$$
$$=\sin\{(90^\circ-A)-(-B)\}=\sin(90^\circ-A)\cos(-B)-\cos(90^\circ-A)\sin(-B)$$
$$=\cos A(\cos B)-\sin A(-\sin B)$$
$$=\cos A\cos B+\sin A\sin B.$$

10. Given $\cos(A+B)=\cos A\cos B-\sin A\sin B$:

then $$\sin(A+B)=\cos\{90^\circ-(A+B)\}=\cos\{(90^\circ-A)-B\}$$
$$=\cos\{(90^\circ-A)+(-B)\}=\cos(90^\circ-A)\cos(-B)-\sin(90^\circ-A)\sin(-B)$$
$$=\sin A(\cos B)-\cos A(-\sin B)$$
$$=\sin A\cos B+\cos A\sin B.$$

11. Here let $BAC = A$, $CAD = B$, $BAD = A - B$:



$$\therefore \sin(A-B) = \sin BAD = \frac{PM}{AP} = \frac{QN+PR}{AP}$$

$$= \frac{QN}{AQ}\cdot\frac{AQ}{AP} + \frac{PR}{PQ}\cdot\frac{PQ}{AP}:$$

now $\dfrac{QN}{AQ} = \sin BAC = \sin A$, $\dfrac{AQ}{AP} = \cos CAD = \cos B$, $\dfrac{PQ}{AP} = \sin B$,

$$\frac{PR}{PQ} = \sin PQR = \cos RQA = \cos CAB = \cos(180^\circ - A) = -\cos A;$$

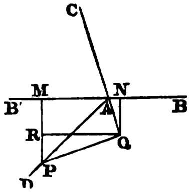
$\therefore$ $\sin(A-B) = \sin A \cos B + (-\cos A)\sin B = \sin A \cos B - \cos A \sin B$.

So $\cos(A-B) = \dfrac{-AM}{AP}$ (using AM as in (44) to denote merely the *magnitude* of the line)

$$= -\frac{AN-QR}{AP} = -\frac{AN}{AQ}\cdot\frac{AQ}{AP} + \frac{QR}{PQ}\cdot\frac{PQ}{AP}$$

$$= -\cos(180^\circ - A)\cos B + \sin(180^\circ - A)\sin B = \cos A \cos B + \sin A \sin B.$$

12. Here let $BAC = A$, $CAD = B$; $\therefore \sin(A+B) = \sin BAD = \dfrac{-PM}{AP}$



$$= -\frac{QN+PR}{AP} = -\frac{QN}{AQ}\cdot\frac{AQ}{AP} - \frac{PR}{PQ}\cdot\frac{PQ}{AP}:$$

now $\dfrac{QN}{AQ} = \sin BAQ = \sin(180^\circ - A) = \sin A$,

$\dfrac{AQ}{AP} = \cos PAQ = \cos(180^\circ - B) = -\cos B$, $\dfrac{PQ}{AP} = \sin B$,

$$\frac{PR}{PQ} = \sin PQR = \cos RQA = \cos QAB = \cos(180^\circ - A) = -\cos A;$$

$$\therefore \sin(A+B) = -\sin A(-\cos B) - (-\cos A)\sin B$$

$$= \sin A \cos B + \cos A \sin B.$$

So $\cos(A+B) = \dfrac{-AM}{AP} = -\dfrac{QR-AN}{AP} = -\dfrac{QR}{PQ}\cdot\dfrac{PQ}{AP} + \dfrac{AN}{AQ}\cdot\dfrac{AQ}{AP}$

$$= -\cos PQR . \sin PAQ + \cos BAQ \cos PAQ;$$

and $\cos PQR = \sin RQA = \sin QAB = \sin(180^\circ - A) = \sin A$,

$\sin PAQ = \sin(180^\circ - B) = \sin B$, $\cos BAQ = \cos(180^\circ - A) = -\cos A$,

$\cos PAQ = \cos(180^\circ - B) = -\cos B$;

$$\therefore \cos(A+B) = -\sin A \sin B + (-\cos A)(-\cos B)$$

$$= \cos A \cos B - \sin A \sin B.$$

Ex. 11.

1. $\cos A = \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A = (\cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A)\,(\cos^2 \tfrac{1}{2}A + \sin^2 \tfrac{1}{2}A)$.

2. $$\cot A - \tan A = \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A} = \frac{\cos^2 A - \sin^2 A}{\sin A \cos A} = \frac{\cos 2A}{\frac{1}{2}\sin 2A} = \text{\&c.}$$

3. $$\tan \tfrac{1}{2}A = (74)\ \frac{\sin A}{1 + \cos A} = \frac{\sin A}{\cos A} \div \left(\frac{1}{\cos A} + 1\right) = \frac{\tan A}{\sec A + 1}.$$

4. $$\operatorname{cosec} 2A = \frac{1}{\sin 2A} = \frac{1}{2 \sin A \cos A} = \frac{1}{2 \sin^2 A \, \dfrac{\cos A}{\sin A}} = \frac{\operatorname{cosec}^2 A}{2 \cot A}.$$

5. $$\sec A = 1 + (\sec A - 1) = 1 + \left(\frac{1}{\cos A} - 1\right) = 1 + \frac{1 - \cos A}{\cos A}$$
$$= 1 + \frac{\sin A}{\cos A} \cdot \frac{1 - \cos A}{\sin A} = 1 + \tan A \tan \tfrac{1}{2}A, \text{ by (74).}$$

6. $$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{\sin^2 \frac{1}{2}A + \cos^2 \frac{1}{2}A}{2 \sin \frac{1}{2}A \cos \frac{1}{2}A} = \frac{1}{2}\left(\frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} + \frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}A}\right) = \text{\&c.}$$

7. $$u = \frac{2 \sin A + 2 \sin A \cos A}{2 \sin A - 2 \sin A \cos A} = \frac{1 + \cos A}{1 - \cos A} = \cot^2 \tfrac{1}{2}A, \text{ by (76).}$$

8. $$\text{By (74)}, \ u = \tan A \times \frac{\cos A}{1 + \cos A} = \frac{\sin A}{1 + \cos A} = \tan \tfrac{1}{2}A, \text{ by (74).}$$

9. $$\text{By (73)}, \ u = \frac{(\cos \frac{1}{2}A + \sin \frac{1}{2}A)^2}{2 \cos^2 \frac{1}{2}A} = \frac{1}{2}\left(1 + \frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A}\right)^2 = \text{\&c.}$$

10. $$u = \frac{1}{\sin 2A + 1} = (73)\ \frac{\cos^2 A + \sin^2 A}{(\cos A + \sin A)^2} = \frac{1 + \tan^2 A}{(1 + \tan A)^2}.$$

11. $$u = \frac{\cos A}{\sin A} - \frac{\cos 2A}{\sin 2A} = \frac{\sin 2A \cos A - \cos 2A \sin A}{\sin A \sin 2A}$$
$$= \frac{\sin (2A - A)}{\sin A \sin 2A} = \frac{1}{\sin 2A}.$$

12. $$u = \frac{\sin \frac{1}{2}A + 1}{\cos \frac{1}{2}A} = (73)\ \frac{(\cos \frac{1}{4}A + \sin \frac{1}{4}A)^2}{\cos^2 \frac{1}{4}A - \sin^2 \frac{1}{4}A} = \frac{\cos \frac{1}{4}A + \sin \frac{1}{4}A}{\cos \frac{1}{4}A - \sin \frac{1}{4}A}$$
$$= \frac{1 + \tan \frac{1}{4}A}{1 - \tan \frac{1}{4}A} = \tan (45^\circ + \tfrac{1}{4}A).$$

Ex. 12.

1. $2 \sin 30^\circ \cos A = \cos A$. 2. $2 \sin 30^\circ \sin \tfrac{1}{2}A = \sin \tfrac{1}{2}A$.

3. $\sin^2 45^\circ - \sin^2 A = \tfrac{1}{2} - \sin^2 A = \tfrac{1}{2}(1 - 2 \sin^2 A) = \tfrac{1}{2} \cos 2A$.

4. $\cos^2 60^\circ - \sin^2 A = \tfrac{1}{4} - \sin^2 A = \tfrac{1}{4}(1 - 4 \sin^2 A) = \tfrac{1}{4}\{1 - 2(1 - \cos 2A)\}$.

5. $1 + (\cos^2 2A - \sin^2 2B) = \cos^2 2A + \cos^2 2B$.

6. $u = \cos \{\tfrac{3}{2}(A - B) - \tfrac{1}{2}(A + B)\} - \cos \{\tfrac{3}{2}(A - B) + \tfrac{1}{2}(A + B)\}$.

7. $u = 4\sin A\,(\sin^2 60^\circ - \sin^2 A) = 4\sin A\,(\frac{3}{4} - \sin^2 A)$
$= 3\sin A - 4\sin^3 A = \sin 3A.$

8. $(\sin^2 A - \sin^2 B) + (\sin^2 B - \sin^2 C) + (\sin^2 C - \sin^2 A) = 0.$

9. $(\cos^2 A - \sin^2 2B) + (\sin^2 2B - \sin^2 C) = \cos^2 A - \sin^2 C$
$= \cos(A + C)\cos(A - C).$

10. $u = \frac{1}{2}\{\cos(A - B - C) - \cos(A - B + C) + \cos(B - C - A)$
$- \cos(B - C + A) + \cos(C - A - B) - \cos(C - A + B)\} = 0,$
since the first and sixth, second and third, fourth and fifth, terms destroy each other.

11. $u = \frac{1}{2}\{\sin(A + A + B) + \sin(A - A - B)\}$
$- \frac{1}{2}\{\sin(A - B + A) + \sin(A - B - A)\}$
$= \frac{1}{2}\{\sin(2A + B) - \sin B\} - \frac{1}{2}\{\sin(2A - B) - \sin B\}$
$= \frac{1}{2}\{\sin(2A + B) - \sin(2A - B)\} = \frac{1}{2}\{2\cos 2A\,\sin B\}.$

12. $\{1 - \cos(A + B)\}\{1 - \cos(A - B)\}$
$= 1 - \{\cos(A + B) + \cos(A - B)\} + \cos(A + B)\cos(A - B)$
$= 1 - 2\cos A\cos B + \cos^2 B - \sin^2 A = (\cos A - \cos B)^2.$

Ex. 13.

1. $\sin 5A\,\sin A = \frac{1}{2}\{\cos(5A - A) - \cos(5A + A)\} = \&c,$
$\text{or} = \sin^2\frac{1}{2}(5A + A) - \sin^2\frac{1}{2}(5A - A) = \&c:$
$\sin 3A\,\sin 2A = \frac{1}{2}\{\cos(3A - 2A) - \cos(3A + 2A)\} = \&c,$
$\text{or} = \sin^2\frac{1}{2}(3A + 2A) - \sin^2\frac{1}{2}(3A - 2A) = \&c:$
$\sin A\,\sin\frac{1}{2}A = \frac{1}{2}\{\cos(A - \frac{1}{2}A) - \cos(A + \frac{1}{2}A)\} = \&c,$
$\text{or} = \sin^2\frac{1}{2}(A + \frac{1}{2}A) - \sin^2\frac{1}{2}(A - \frac{1}{2}A) = \&c.$

2. $\cos 3A\,\cos\frac{1}{2}A = \frac{1}{2}\{\cos(3A + \frac{1}{2}A) + \cos(3A - \frac{1}{2}A)\} = \&c,$
$\text{or} = \cos^2\frac{1}{2}(3A + \frac{1}{2}A) - \sin^2\frac{1}{2}(3A - \frac{1}{2}A) = \&c:$
$\cos 2A\,\cos A = \frac{1}{2}\{\cos(2A + A) + \cos(2A - A)\} = \&c,$
$\text{or} = \cos^2\frac{1}{2}(2A + A) - \sin^2\frac{1}{2}(2A - A) = \&c:$
$\cos\frac{1}{2}A\,\cos\frac{1}{4}A = \frac{1}{2}\{\cos(\frac{1}{2}A + \frac{1}{4}A) + \cos(\frac{1}{2}A - \frac{1}{4}A)\} = \&c,$
$\text{or} = \cos^2\frac{1}{2}(\frac{1}{2}A + \frac{1}{4}A) - \sin^2\frac{1}{2}(\frac{1}{2}A - \frac{1}{4}A) = \&c.$

3. $\sin 2A\,\cos 3A = \frac{1}{2}\{\sin(2A + 3A) + \sin(2A - 3A)\} = \&c,$
$\text{or} = \cos(90^\circ - 2A)\cos 3A$
$= \cos^2\frac{1}{2}(90^\circ - 2A + 3A) - \sin^2\frac{1}{2}(90^\circ - 2A - 3A) = \&c:$
$\sin\frac{1}{2}A\,\cos\frac{5}{2}A = \frac{1}{2}\{\sin(\frac{1}{2}A + \frac{5}{2}A) + \sin(\frac{1}{2}A - \frac{5}{2}A)\} = \&c,$
$\text{or} = \cos(90^\circ - \frac{1}{2}A)\cos\frac{5}{2}A$
$= \cos^2\frac{1}{2}(90^\circ - \frac{1}{2}A + \frac{5}{2}A) - \sin^2\frac{1}{2}(90^\circ - \frac{1}{2}A - \frac{5}{2}A) = \&c:$
$\cos\frac{3}{2}A\,\sin A = \frac{1}{2}\{\sin(A + \frac{3}{2}A) + \sin(A - \frac{3}{2}A)\} = \&c,$
$\text{or} = \sin(90^\circ - \frac{3}{2}A)\sin A$
$= \sin^2\frac{1}{2}(90^\circ - \frac{3}{2}A + A) - \sin^2\frac{1}{2}(90^\circ - \frac{3}{2}A - A) = \&c.$

4. $\sin(A+B)\sin C = \frac{1}{2}\{\cos(A+B-C) - \cos(A+B+C)\}$,
$\quad\text{or } = \sin^2\frac{1}{2}(A+B+C) - \sin^2\frac{1}{2}(A+B-C)$:
$\cos(A-B)\cos C = \frac{1}{2}\{\cos(A-B+C) + \cos(A-B-C)\}$,
$\quad\text{or } = \cos^2\frac{1}{2}(A-B+C) - \sin^2\frac{1}{2}(A-B-C)$:
$u = \frac{1}{2}\{\sin(A+B+A-B) + \sin(A+B-A+B)\} = \&c$,
$\text{or } = \cos(90^\circ - A - B)\cos(A-B)$
$= \cos^2\frac{1}{2}(90^\circ - A - B + A - B) - \sin^2\frac{1}{2}(90^\circ - A - B - A + B) = \&c.$

5. $u = 2\cos\frac{1}{2}\{A + (A+2B)\}\cos\frac{1}{2}\{A - (A+2B)\} = \&c.$

6. $u = \cos^2\frac{1}{2}A\,\{1 + \tan\frac{1}{2}A) + \sec\frac{1}{2}A\}\,\{(1+\tan\frac{1}{2}A) - \sec\frac{1}{2}A\}$
$= \cos^2\frac{1}{2}A\,\{(1+\tan\frac{1}{2}A)^2 - \sec^2\frac{1}{2}A\}$
$= \cos^2\frac{1}{2}A\,\{(1 + 2\tan\frac{1}{2}A + \tan^2\frac{1}{2}A) - (1 + \tan^2\frac{1}{2}A)\}$
$= \cos^2\frac{1}{2}A \times 2\tan\frac{1}{2}A = 2\sin\frac{1}{2}A\cos\frac{1}{2}A = \sin A.$

7. $u = \dfrac{\sin\frac{3}{2}A\,\sin\frac{1}{2}A}{\cos\frac{3}{2}A\,\cos\frac{1}{2}A} = \dfrac{\frac{1}{2}\{\cos(\frac{3}{2}A - \frac{1}{2}A) - \cos(\frac{3}{2}A + \frac{1}{2}A)\}}{\frac{1}{2}\{\cos(\frac{3}{2}A - \frac{1}{2}A) + \cos(\frac{3}{2}A + \frac{1}{2}A)\}} = \&c.$

8. $u = \dfrac{\sin^2 2A}{\cos^2 2A} - \dfrac{\sin^2 A}{\cos^2 A} = \dfrac{\sin^2 2A\cos^2 A - \sin^2 A\cos^2 2A}{\cos^2 2A\cos^2 A}$
$=$, as in (80), $\dfrac{\sin(2A+A)\sin(2A-A)}{\cos^2 2A\cos^2 A} = \&c.$

9. $u = \dfrac{\sin 2A}{\cos 2A} - \dfrac{\sin A}{\cos A} = \dfrac{\sin 2A\cos A - \sin A\cos 2A}{\cos 2A\cos A}$
$= \dfrac{\sin(2A-A)}{\cos 2A\cos A} = \dfrac{\sin A}{\frac{1}{2}\{\cos(2A+A) + \cos(2A-A)\}} = \&c.$

0. $u = \dfrac{\sin A}{\cos A} = \dfrac{2\sin A\cos\frac{1}{3}A}{2\cos A\cos\frac{1}{3}A} = \dfrac{\sin(A+\frac{1}{3}A) + \sin(A-\frac{1}{3}A)}{\cos(A+\frac{1}{3}A) + \cos(A-\frac{1}{3}A)} = \&c.$

1. $u = \dfrac{\sin 2A}{\cos 2A} = \dfrac{2\sin 2A\cos A}{2\cos 2A\cos A} = \dfrac{\sin(2A+A) + \sin(2A-A)}{\cos(2A+A) + \cos(2A-A)} = \&c.$

2. $u = \dfrac{\sin 3A}{\cos 3A} = \dfrac{\sin 3A(1 + 2\cos 2A)}{\cos 3A(1 + 2\cos 2A)} = \dfrac{\sin 3A + 2\sin 3A\cos 2A}{\cos 3A + 2\cos 3A\cos 2A}$
$= \dfrac{\sin 3A + \sin(3A+2A) + \sin(3A-2A)}{\cos 3A + \cos(3A+2A) + \cos(3A-2A)} = \&c.$

Ex. 14.

1. $u = 2\sin\frac{1}{2}(5A+3A)\cos\frac{1}{2}(5A-3A) = \&c$;
$u = 2\sin\frac{1}{2}(\frac{1}{2}A + A)\sin\frac{1}{2}(\frac{1}{2}A - A) = \&c.$

2. $u = 2\sin\frac{1}{2}\{A + (A-2B)\}\cos\frac{1}{2}\{A - (A-2B)\} = \&c$;
$u = \sin(90^\circ - A) - \sin(A-B)$
$= 2\sin\frac{1}{2}(90^\circ - A - A + B)\cos\frac{1}{2}(90^\circ - A + A - B) = \&c.$

3. $u = 2\sin\frac{1}{2}\{(A+B)-B\}\cos\frac{1}{2}\{(A+B)+B\} =$ &c;
$u = 2(\sin\frac{1}{2}A + \sin\frac{1}{4}A) = 2\times 2\sin\frac{1}{2}(\frac{1}{2}A+\frac{1}{4}A)\cos\frac{1}{2}(\frac{1}{2}A-\frac{1}{4}$

4. $u = (1-\cos 2A)-(1-\cos A) = \cos A - \cos 2A$
$= 2\sin\frac{1}{2}(2A+A)\sin\frac{1}{2}(2A-A) =$ &c;
$u = 2\cos\frac{1}{2}(A-B+B-C)\cos\frac{1}{2}(A-B-B+C) =$ &c.

5. $u = \sin(A+A-2B)\sin(A-A+2B) =$ &c;
$u = \sin(\frac{1}{2}A+A)\sin(\frac{1}{2}A-A) =$ &c.

6. $u = 2\sin\frac{1}{2}\{(A-2B)-(2A+B)\}\sin\frac{1}{2}\{(A-2B)+(2A+B)\} =$
$u = 2\sin\frac{1}{2}(A+2A)\cos\frac{1}{2}(A-2A) =$ &c.

7. $u = \{1-\cos(A+C)\}-\{1-\cos(B+C)\} = \cos(B+C)-\cos(A$
$= 2\sin\frac{1}{2}(A+C+B+C)\sin\frac{1}{2}(A+C-B-C) =$ &c;
$u = \sin\{(A-2B)+B\}\sin\{(A-2B)-B\} =$ &c.

8. $u = 4\{\sin^2\frac{1}{2}A - \sin^2\frac{1}{2}(A-B)\}$
$= 4\sin\{\frac{1}{2}A+\frac{1}{2}(A-B)\}\sin\{\frac{1}{2}A-\frac{1}{2}(A-B)\} =$ &c;
$u = \cos(90^\circ - A) + \cos A$
$= 2\cos\frac{1}{2}(90^\circ - A + A)\cos\frac{1}{2}(90^\circ - A - A) =$ &c.

9. $u = -\{\cos^2(A+B)-\sin^2(A+B)\} = -\cos 2(A+B)$ by (72);
$u = \cos\{(A-B)+(A+B)\}\cos\{(A-B)-(A+B)\} =$ &c.

10. $u = \cos(90^\circ - A - B) + \cos(A-B)$
$= 2\cos\frac{1}{2}(90^\circ - A - B + A - B)\cos\frac{1}{2}(90^\circ - A - B - A + B) =$ &c;
$u = \cos(90^\circ - \frac{1}{2}A) - \cos\frac{1}{2}A$
$= -2\sin\frac{1}{2}(90^\circ - \frac{1}{2}A + \frac{1}{2}A)\sin\frac{1}{2}(90^\circ - \frac{1}{2}A - \frac{1}{2}A) =$ &c.

Ex. 15.

1. $u = \sin(180^\circ - C)\sin(180^\circ - A) = \sin C \sin A.$

2. $\cot A = -\cot(B+C) = -\dfrac{1}{\tan(B+C)} = -\dfrac{\tan B\tan C - 1}{\tan B + \tan C}$
$= -\dfrac{1-\cot B\cot C}{\cot C + \cot B}$, by dividing each term by $\tan B\tan C$:
hence $\cot A\cot C + \cot A\cot B = -1 + \cot B\cot C.$

Or thus: since in this case, by (85), we have
$$\tan A + \tan B + \tan C = \tan A\tan B\tan C,$$
dividing each term of this result by $\tan A\tan B\tan C$, we get
$$\cot B\cot C + \cot A\cot C + \cot A\cot B = 1.$$

3. $u = \sin 180^\circ + \sin(180^\circ - 2C) + \sin(180^\circ - 2B) + \sin(180^\circ - 2A)$
$= \sin 2C + \sin 2B + \sin 2A = \sin 2C + 2\sin(A+B)\cos(A-B)$
$= 2\sin(A+B)\{\cos(A-B) - \cos(A+B)\},$
(for $\sin 2C = 2\sin C\cos C$, $\sin C = \sin A+B)$, $\cos C = -\cos(A+B)$,)
$= 2\sin C\{2\sin A\sin B\} =$ &c.

$u = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) + \sin C$

$= 2\sin\frac{1}{2}(A+B)\{\cos\frac{1}{2}(A-B) + \cos\frac{1}{2}(A+B)\}$,

or $\sin C = 2\sin\frac{1}{2}C\cos\frac{1}{2}C$, $\sin\frac{1}{2}C = \cos\frac{1}{2}(A+B)$, $\cos\frac{1}{2}C = \sin\frac{1}{2}(A+B)$,)

$= 2\cos\frac{1}{2}C\{2\cos\frac{1}{2}A\cos\frac{1}{2}B\} =$ &c.

5. $u = 1 + (\cos^2 A - \sin^2 B) + \cos^2 C + \cos C\{\cos(A+B) + \cos(A-B)\}$

$= 1 + \cos(A+B)\cos(A-B) + \cos^2 C + \cos C\{-\cos C + \cos(A-B)\}$

$= 1$, since $\cos(A+B) = -\cos C$.

6. $u = 1 - (\cos^2\frac{1}{2}A - \sin^2\frac{1}{2}B) + \sin^2\frac{1}{2}C$

$+ \sin\frac{1}{2}C\{\cos\frac{1}{2}(A-B) - \cos\frac{1}{2}(A+B)\}$

$= 1 - \cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) + \sin^2\frac{1}{2}C + \sin\frac{1}{2}C\{\cos\frac{1}{2}(A-B) - \sin\frac{1}{2}C\}$

$= 1$, since $\cos\frac{1}{2}(A+B) = \sin\frac{1}{2}C$.

7. $\cot A = \tan(B+C) = \dfrac{\tan B + \tan C}{1 - \tan B\tan C} = \dfrac{\cot C + \cot B}{\cot B\cot C - 1}$,

dividing each term by $\tan B\tan C$;

$$\text{hence } \cot A\cot B\cot C - \cot A = \cot C + \cot B.$$

Or thus: since in this case, by (85), we have

$$\tan A\tan B + \tan A\tan C + \tan B\tan C = 1,$$

dividing each term of this result by $\tan A\tan B\tan C$, we get

$$\cot C + \cot B + \cot A = \cot A\cot B\cot C.$$

8. $\sin(A+B+C) = \sin 90° = 1 = \sin(A+B)\cos C + \cos(A+B)\sin C$

$= (\sin A\cos B + \cos A\sin B)\cos C + (\cos A\cos B - \sin A\sin B)\sin C$:

$\therefore \cos B\cos C\sin A + \cos A\cos C\sin B + \cos A\cos B\sin C$

$= \sin A\sin B\sin C + 1$,

or, dividing each term by $\cos A\cos B\cos C$,

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C + \sec A\sec B\sec C.$$

$u = 2\sin(A+B)\cos(A-B) + \sin 2C$

$= 2\cos C\{\cos(A-B) + \cos(A+B)\}$,

(for $\sin(A+B) = \sin(90° - C) = \cos C$, and $\sin C = \cos(A+B)$,)

$= 2\cos C\{2\cos A\cos B\}$.

$u = \cos(90° + B - C) + \cos(90° + C - A) + \cos(90° + A - B)$

$= \sin(C-B) + \sin(A-C) + \sin(B-A)$

$= 2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B-2C) - \sin(A-B)$

$= 2\sin\frac{1}{2}(A-B)\{\cos\frac{1}{2}(A+B-2C) - \cos\frac{1}{2}(A-B)\}$

$= 2\sin\frac{1}{2}(A-B)\{2\sin\frac{1}{2}(A-C)\sin\frac{1}{2}(C-B)\} =$ &c.

$\cos\{(B+C)+A\} + \cos\{(B+C)-A\} = 2\cos(B+C)\cos A$,

$\cos\{A-(B-C)\} + \cos\{A+(B-C)\} = 2\cos(B-C)\cos A$;

and $2\cos A\{\cos(B+C) + \cos(B-C)\} = 2\cos A \cdot 2\cos B\cos C$.

12. $u = \sin(A+B)\sin B\cos C + \cos B\sin C(\sin A\cos B + \cos A\sin B)$
$= \sin(A+B)\sin B\cos C + \sin A\sin C(1-\sin^2 B)$
$+ \cos A\sin B\cos B\sin C$
$= \sin B\{\sin(A+B)\cos C + \sin C(\cos A\cos B - \sin A\sin B)\}$
$+ \sin A\sin C$
$= \sin B\{\sin(A+B)\cos C + \sin C\cos(A+B)\} + \sin A\sin C$
$= \sin B\sin(A+B+C) + \sin A\sin C.$

Ex. 16.

1. $2\tan^{-1}\frac{1}{2} = \tan^{-1}\dfrac{\frac{2}{2}}{1-\frac{1}{4}} = \tan^{-1}\dfrac{1}{\frac{3}{4}} = \tan^{-1}\frac{4}{3}.$

2. $2\tan^{-1}\frac{1}{3} = \tan^{-1}\dfrac{\frac{2}{3}}{1-\frac{1}{9}} = \tan^{-1}\frac{3}{4} = \cot^{-1}\frac{4}{3}.$

3. $\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{3}{4}$ (by the preceding Ex.)

$$= \tan^{-1}\dfrac{\frac{1}{7}+\frac{3}{4}}{1-\frac{3}{28}} = \tan^{-1}1 = \tfrac{1}{4}\pi.$$

4. $\cot^{-1}\frac{3}{4} + \cot^{-1}\frac{1}{7} = \tan^{-1}\frac{4}{3} + \tan^{-1}7 = \tan^{-1}\dfrac{\frac{4}{3}+7}{1-\frac{28}{3}} = \tan^{-1}(-1) = \frac{3}{4}\pi.$

5. Let $\theta = \sin^{-1}\sqrt{\frac{3}{5}}$; $\therefore \sin\theta = \sqrt{\frac{3}{5}}$, and $\cos\theta = \sqrt{\frac{2}{5}}$; and $\tan\theta = \sqrt{\frac{3}{2}}$.

6. Let $\theta = \sin^{-1}\frac{3}{5}$; $\therefore \sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, and $\tan\theta = \frac{3}{4}$;

hence $u = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{4} = \tan^{-1}\dfrac{\frac{3}{4}+\frac{5}{4}}{1-\frac{15}{16}} = \tan^{-1}32.$

7. Let $\theta = \sin^{-1}\sqrt{\frac{1}{5}}$; $\therefore \sin\theta = \dfrac{1}{\sqrt{5}}$, $\cos\theta = \dfrac{2}{\sqrt{5}}$, and $\tan\theta = \frac{1}{2}$;

hence $u = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} = \tan^{-1}1 = \frac{1}{4}\pi.$

8. $u = \tan^{-1}\dfrac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}} + \tan^{-1}\dfrac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}} = \tan^{-1}\frac{4}{7} + \tan^{-1}\frac{3}{11} = \tan^{-1}\dfrac{\frac{4}{7}+\frac{3}{11}}{1-\frac{12}{77}}$
$= \tan^{-1}1 = \frac{1}{4}\pi.$

9. $u = \frac{1}{2}\sqrt{(2+2\cos 45^\circ)}$ by (76) $= \frac{1}{2}\sqrt{(2+\sqrt{2})}.$

10. $u = \frac{1}{2}\sqrt{(2+2\cos 135^\circ)} = \frac{1}{2}\sqrt{(2-\sqrt{2})}.$

11. $u = \sqrt{\dfrac{1-\cos 75^\circ}{1+\cos 75^\circ}}$ by (76) $= \sqrt{\dfrac{1-\dfrac{\sqrt{3}-1}{2\sqrt{2}}}{1+\dfrac{\sqrt{3}-1}{2\sqrt{2}}}}$ by (71 Ex. 1)
$= \sqrt{\dfrac{2\sqrt{2}-\sqrt{3}+1}{2\sqrt{2}+\sqrt{3}-1}}.$

12. $u = \frac{1}{2}\sqrt{(2-2\cos 15^\circ)}$ by (76) $= \frac{1}{2}\sqrt{(2-\dfrac{\sqrt{3}+1}{\sqrt{2}})}$ by (71 Ex. 1)
$= \frac{1}{2}\sqrt{\dfrac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{2}}} = \dfrac{\sqrt{(2\sqrt{2}-\sqrt{3}-1)}}{2\sqrt[4]{2}}.$

3. $u = 1 - \cos 15° = 1 - \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \&c.$

4. $u = \frac{1}{2}\sqrt{(2+2\cos 22\frac{1}{2}°)} = \frac{1}{2}\sqrt{\{2+\sqrt{(2+2\cos 45°)}\}}$ by (76)
$= \frac{1}{2}\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}.$

5. $u = \cos 33° \, 45' = \frac{1}{2}\sqrt{(2+2\cos 67\frac{1}{2}°)} = \frac{1}{2}\sqrt{\{2+\sqrt{(2+2\cos 135°)}\}}$
$= \frac{1}{2}\sqrt{\{2+\sqrt{(2-\sqrt{2})}\}}.$

6. $u = 2\sin 67\frac{1}{2}° = \sqrt{(2-2\cos 135°)} = \sqrt{(2+\sqrt{2})}.$

Ex. 17.

1. Let h, h', be the heights of the towers, d, d', their distances from the observer, θ, θ', the angles they subtend to his eye: then $\sin\theta = \theta = \dfrac{h}{d}$, $\sin\theta' = \theta' = \dfrac{h'}{d'}$;

$$\therefore \frac{h}{h'} = \frac{\theta}{\theta'} \times \frac{d}{d'} = \frac{5193''}{5770''} \times \frac{2\frac{1}{2}}{3\frac{1}{2}} = \frac{9}{10} \times \frac{5}{7} = \frac{9}{14}.$$

2. Regarding the Earth's radius as a small circular arc, we have $a = \theta r$, or r (the Moon's distance) $= \dfrac{a}{\theta} = a \times \dfrac{57°.29577}{57' \, 1''.8}$
$= a \times \dfrac{57°.29577}{0°.9505} = a \times 60$ nearly $= 240{,}000$ miles.

3. As in (Ex. 17. 2), $a = \theta r = \dfrac{31' \, 15''}{57°.29577} \times r$, or (more simply)
$= \dfrac{31\frac{1}{4}' \times \pi}{180°} \times r = \dfrac{31\frac{1}{4}}{180 \times 60} \times \dfrac{22}{7} \times 240{,}000 = \dfrac{137500}{63} = 2182.5$ miles.

Now, the distances of the Earth from the Moon and of the Moon from the Earth being the same, their disks will appear to each other as circles, whose diameters will be proportional to the actual diameters of the Earth and Moon, that is, will be in the ratio of 8000 : 2182.5, or nearly of 4 : 1; hence the areas of these circles will be as 16 : 1.

$a = \theta r$, or r (the star's distance) $= \dfrac{a}{\theta} = \dfrac{96{,}000{,}000}{\frac{1}{206265}}$
$= 19{,}801{,}440{,}000{,}000 = 20$ billions of miles nearly.

The dip is 8 inches for one mile, and varies as the square of the distance; $\therefore$ for 3 miles, the dip $= 9 \times 8 = 72$ inches $= 6$ feet or 1 fathom.

$h : h' :: d^2 : d'^2$; $\therefore$ 8 *in* : $2\frac{1}{2}$ *m* :: $1 : d'^2$; hence we get
$d'^2 = \dfrac{2\frac{1}{2} \times 1760 \times 3 \times 12}{8} = 19800$, and $d' = 140.7 = 141$ miles.

7. $d^2 = \dfrac{28000 \times 12}{8} = 42000$, and $d' = 204.9 = 205$ miles.

8. $d^2 = 2rh$; $\therefore$ (observing that 6600 feet $= 1\frac{1}{4}$ mile) we have $r = \dfrac{10000}{2\frac{1}{2}} = 4000$ miles.

9. Let $d = 4$ miles $=$ the distance of each point from the middle point between them, where the visual ray, by which each is seen from the other, will just graze the Earth's surface; then $d^2 = 2rh$, and (observing that 10 feet $= \frac{1}{528}$ mile) we have $2r = \dfrac{16}{\frac{1}{528}} = 8448$ miles.

10. $h : h' :: d^2 : d'^2$; $\therefore$

(1) 8 in : 40 ft :: 1 : d'^2, or $d'^2 = \dfrac{40 \times 12}{8} = 60$, and $d' = 7.7$ miles;

(2) 8 in : 120 ft :: 1 : d'^2, or $d'^2 = 180$, and $d' = 13.4$ miles.

Ex. 12.

1. (i) $D = 2644$;

$\therefore d = \frac{15}{60} \times 2644 = 661$, and $\sin 24^\circ\, 37'\, 15'' = .4166114$:

(ii) $d = 834$; $\therefore n = \frac{834}{2644} \times 60 = 18.9 = 19$, and $A' = 24^\circ\, 37'\, 19''$.

2. (i) $D = -2701$;

$\therefore d = \frac{24}{60} \times -2701 = -1080$, and $\cos 68^\circ\, 12'\, 24'' = .3712598$:

(ii) $d = -1679$; $\therefore n = \frac{1679}{2701} \times 60 = 37$, and $A' = 68^\circ\, 12'\, 37''$.

3. (i) $\tan 45^\circ = 1 = 1.0000000$; $\therefore D = 5819$, and $d = \frac{35}{60} \times 5819 = 3394$, and $\tan 45^\circ\, 0'\, 35'' = 1.0003394$:

(ii) $d = 2345$; $\therefore n = \frac{2345}{5819} \times 60 = 24$, and $A' = 45^\circ\, 0'\, 24''$.

4. (i) $\cot 30^\circ = \sqrt{3} = 1.7320508$; $\therefore D = -11630$, and $d = \frac{17}{60} \times -11630 = -3295$, and $\cot 30^\circ\, 0'\, 17'' = 1.7317213$:

(ii) $d = -8385$; $\therefore n = \frac{8385}{11630} \times 60 = 43$, and $A' = 30^\circ\, 0'\, 43''$.

5. (i) $\sec 60^\circ = 2.0000000$; $\therefore D = 10071$, and $d = \frac{25}{60} \times 10071 = 4196$, and $\sec 59^\circ\, 59'\, 25'' = 1.9994125$:

(ii) $d = 749$; $\therefore n = \frac{749}{10071} \times 60 = 4$, and $A' = 59^\circ\, 59'\, 4''$.

6. (i) $\operatorname{cosec} 60^\circ = \frac{2}{3}\sqrt{3} = 1.1547005$; $\therefore D = -1938$, and $d = \frac{37}{60} \times -1938 = -1195$, and $\operatorname{cosec} 60^\circ\, 0'\, 37'' = 1.1545810$:

(ii) $d = -980$; $\therefore n = \frac{980}{1938} \times 60 = 30$, and $A' = 60^\circ\, 0'\, 30''$.

7. $\{1 - \cos(A - B)\} \times \{1 + \cos(A + B)\} = 1 - \{\cos(A - B) - \cos(A + B)\}$
$- \cos(A + B)\cos(A - B) = 1 - 2\sin A \sin B - (\cos^2 B - \sin^2 A)$
$= \sin^2 A - 2\sin A \sin B + \sin^2 B = (\sin A - \sin B)^2$.

8. $u = \dfrac{1+\tan\frac{1}{2}A}{1-\tan\frac{1}{2}A} + \dfrac{1-\tan\frac{1}{2}A}{1+\tan\frac{1}{2}A} = \dfrac{(1+\tan\frac{1}{2}A)^2 + (1-\tan\frac{1}{2}A)^2}{1-\tan^2\frac{1}{2}A}$

$= \dfrac{2+2\tan^2\frac{1}{2}A}{1-\tan^2\frac{1}{2}A} = \dfrac{2(\cos^2\frac{1}{2}A+\sin^2\frac{1}{2}A)}{\cos^2\frac{1}{2}A-\sin^2\frac{1}{2}A} = \dfrac{2}{\cos A}$.

9. $u = \dfrac{\sin(30^\circ+A)\sin(30^\circ-A)}{\cos(30^\circ+A)\cos(30^\circ-A)} = \dfrac{\cos^2A-\cos^2 30^\circ}{\cos^2A-\sin^2 30^\circ} = \dfrac{\cos^2A-\frac{3}{4}}{\cos^2A-\frac{1}{4}}$

$= \dfrac{2(1+\cos 2A)-3}{2(1+\cos 2A)-1}$.

10. $u = \dfrac{\sin^4A}{\cos^4A} = \dfrac{\sin^2A(1-\cos^2A)}{\cos^2A(1-\sin^2A)} = \dfrac{4\sin^2A\cos^2A-4\sin^2A}{4\sin^2A\cos^2A-4\cos^2A}$

$= \dfrac{\sin^2 2A-4\sin^2A}{\sin^2 2A-4(1-\sin^2A)}$.

1. $1 = \dfrac{\sin^2A}{\sin^2B} + \cos^2A(1-\sin^2C)$, or $\sin^2C\cos^2A = \dfrac{\sin^2A}{\sin^2B} + \cos^2A - 1$

$= \dfrac{\sin^2A}{\sin^2B} - \sin^2A = \dfrac{\sin^2A\cos^2B}{\sin^2B}$; $\therefore \sin C = \dfrac{\sin A}{\cos A} \div \dfrac{\sin B}{\cos B} = \dfrac{\tan A}{\tan B}$.

2. From the given equation we get

$$\sin A + 2\sin A\cos 2B = 2\cos B\sin A,$$

whence $1 + 2\cos 2B = 2\cos B$, or $4\cos^2B - 2\cos B - 1 = 0$;

$\therefore \cos B = \frac{1}{4}(\sqrt{5}+1) = \cos 36^\circ$, or $= -\frac{1}{4}(\sqrt{5}-1) = -\cos 72^\circ$

$= \cos(180^\circ-72^\circ) = \cos 108^\circ$; $\therefore B = 36^\circ$ or 108°,

and, in general terms, $= 2n\pi \pm \frac{1}{5}\pi$ or $2n\pi \pm \frac{3}{5}\pi$.

3. $u = 2\sin\frac{1}{2}(A-C)\cos\frac{1}{2}(A+C) = 2\sin(A-B)\cos B$;

for $A-B = B-C$, and $A-C = A-(2B-A) = 2(A-B)$.

4. $u = \{2\sin\frac{1}{2}(A-C)\cos\frac{1}{2}(A+C)\}\sin B$

$= \{2\sin\frac{1}{2}(A-C)\sin B\}\cos\frac{1}{2}(A+C)$

$= \{2\sin\frac{1}{2}(A-C)\sin\frac{1}{2}(A+C)\}\cos B = (\cos C - \cos A)\cos B$.

Ex. 19.

1. $u = \cos A\cos B + \sin A\sin B = \dfrac{b}{c}\cdot\dfrac{a}{c} + \dfrac{a}{c}\cdot\dfrac{b}{c} =$ &c.

2. $u = \dfrac{1-\tan^2A}{1+\tan^2A}$ (76) $= (1-\dfrac{a^2}{b^2}) \div (1+\dfrac{a^2}{b^2}) =$ &c.

3. $u = \dfrac{2\tan B}{1-\tan^2B} = 2\dfrac{b}{a} \div (1-\dfrac{b^2}{a^2}) =$ &c.

4. $u = \tan\frac{1}{2}(90^\circ-2B) = \tan(45^\circ-B) = \dfrac{1-\tan B}{1+\tan B} = (1-\dfrac{b}{a}) \div (1+\dfrac{b}{a})$.

5. $u = \cos(180^\circ-3B) = -\cos 3B = 3\cos B - 4\cos^3 B$ (75) $= 3\dfrac{a}{c} - 4\dfrac{a^3}{c^3}$.

6. $u = \sqrt{\dfrac{1 - \cos A}{1 + \cos A}}$ (76) $= \sqrt{\{(1 - \dfrac{b}{c}) \div (1 + \dfrac{b}{c})\}}$ = &c.

7. Area $= \frac{1}{2}ab \sin C$ (109) $= \frac{1}{2}ab = \frac{1}{4}\{(a+b)^2 - (a^2 + b^2)\}$
$= \frac{1}{4}\{(a+b)^2 - c^2\}$ = &c.

8. Area $= \frac{1}{2}bc \sin A = \frac{1}{2}(c \cos A) c \sin A = \frac{1}{4}c^2 \sin 2A$.

9. $u = \sin A \cos B - \sin B \cos A = (\dfrac{a}{c} \sin C) \cos B - (\dfrac{b}{c} \sin C) \cos A$
$= (a \cos B - b \cos A) \dfrac{\sin C}{c} = (2ac \cos B - 2bc \cos A) \dfrac{\sin C}{2c^2}$
$=$ by (106) $\{(a^2 + c^2 - b^2) - (b^2 + c^2 - a^2)\} \dfrac{\sin(A+B)}{2c^2}$ = &c.

10. $u = \dfrac{1 - \cos A}{1 - \cos B} = \dfrac{2 \sin^2 \frac{1}{2}A}{2 \sin^2 \frac{1}{2}B} =$ (107) $\dfrac{(s-b)(s-c)}{bc} \times \dfrac{ac}{(s-a)(s-c)}$ = &c.

11. $u = \dfrac{\sin B}{\sin C} \times \dfrac{\cos C}{\cos B} = \dfrac{b}{c} \times \dfrac{a^2 + b^2 - c^2}{2ab} \div \dfrac{a^2 + c^2 - b^2}{2ac}$ = &c.

12. $u = \sin \frac{1}{2}A \cos \frac{1}{2}B - \sin \frac{1}{2}B \cos \frac{1}{2}A$
$= \sqrt{\dfrac{(s-b)(s-c)}{bc}} \sqrt{\dfrac{s(s-b)}{ac}} - \sqrt{\dfrac{(s-a)(s-c)}{ac}} \sqrt{\dfrac{s(s-a)}{bc}}$
$= \dfrac{s-b}{c} \sqrt{\dfrac{s(s-c)}{ab}} - \dfrac{s-a}{c} \sqrt{\dfrac{s(s-c)}{ab}} = \{\dfrac{s-b}{c} - \dfrac{s-a}{c}\} \cdot \sqrt{\dfrac{s(s-c)}{ab}}$ = &c.

13. $u =$ (107) $\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}} = \dfrac{s-c}{s}$ = &c.

14. $u = \{\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}\} \div \{\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} - \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}\}$
$= \{\sqrt{\dfrac{s-b}{s-a}} + \sqrt{\dfrac{s-a}{s-b}}\} \div \{\sqrt{\dfrac{s-b}{s-u}} - \sqrt{\dfrac{s-a}{s-b}}\}$, (divg by $\sqrt{\dfrac{s-c}{s}}$,)
$= \dfrac{(s-b) + (s-a)}{\sqrt{\{(s-a)(s-b)\}}} \div \dfrac{(s-b) - (s-a)}{\sqrt{\{(s-a)(s-b)\}}} = \dfrac{(s-b) + (s-a)}{(s-b) - (s-a)}$ = &c.

15. Area $= \frac{1}{2}bc \sin A = \frac{1}{2}(a^2 - b^2) \dfrac{bc}{a^2 - b^2} \sin A$
$=$ by (110) $\frac{1}{2}(a^2 - b^2) \dfrac{\sin B \sin C}{\sin^2 A - \sin^2 B} \sin A$
$= \frac{1}{2}(a^2 - b^2) \dfrac{\sin B \sin C}{\sin(A+B) \sin(A-B)} \sin A$ = &c, since $\sin C = \sin(A+B)$.

16. Area $= \frac{1}{2}bc \sin A = \dfrac{abc}{2(a+b+c)} \times \dfrac{a+b+c}{a} \sin A$
$=$ by (110) $\dfrac{abc}{2(a+b+c)} \times \dfrac{\sin A + \sin B + \sin C}{\sin A} \sin A$
$= \dfrac{2abc}{a+b+c} \times \dfrac{\sin A + \sin B + \sin C}{4}$ = &c, as in (Ex. 18. 4).

Ex. 20.

1. Area $= \frac{1}{2}bc \sin A = \frac{1}{2} \times 2\frac{2}{3} \times 3\frac{3}{4} \times \sin 30° = \frac{1}{2} \times \frac{8}{3} \times \frac{15}{4} \times \frac{1}{2} = 2\frac{1}{2}$ *sq. yds.*

2. Area $= \frac{1}{2} \times 60 \times 60 \times \sin 60° = 900\sqrt{3} = 1559$ *sq. yds.*

3. $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(50+30+30) = 55, \quad s-a = 5, \quad s-b = 25, \quad s-c = 25;$
$\therefore$ area $= \sqrt{(55 \times 5 \times 25 \times 25)} = 125\sqrt{11} = 415$ *sq. yds.*

4. $s = 150, \quad s - a = 20, \quad s - b = 30, \quad s - c = 100;$
$\therefore$ area $= \sqrt{(150 \times 20 \times 30 \times 100)} = 3000$ *sq. yds;*
and the n° of trees, being 1 to 4 *sq. yds*, $= \frac{1}{4} \times 3000 = 750$.

5. $s = 324, \quad s - a = 108, \quad s - b = 54, \quad s - c = 162;$
$\therefore$ area $= \sqrt{(324 \times 108 \times 54 \times 162)} = \sqrt{(324 \times 108 \times 108 \times 81)}$
$= 18 \times 108 \times 9 = 17496$ *sq. yds* = 3A. 2R. 18P. $11\frac{1}{2}$ *sq. yds.*

6. Area $AEB = \frac{1}{2}AE \cdot AB \sin 45° = \frac{1}{2}AE \cdot 50\sqrt{2}$ *sq. yds*,
which (by question) $= \frac{1}{5}$ of $10000 = 2000$ *sq. yds;*
$\therefore AE = 40\sqrt{2} = 40 \times 1.414 = 57$ *yds.*

7. The parallelogram consists of two equal triangles; the area of each being $\frac{1}{2} \times 28 \times 30 \times \sin 72° = 420 \times \frac{\sqrt{3}+1}{2\sqrt{2}} = 420 \times \frac{\sqrt{6}+\sqrt{2}}{4}$
$= 105 \times (2.449 + 1.414) = 105 \times 3.863 = 405.615$ *sq. ft;*
$\therefore$ area required $= 811$ *sq. ft* $= 90\frac{1}{9}$ *sq. yds.*

8. $s = 200, \quad s - a = 11, \quad s - b = 31, \quad s - c = 158;$
$\therefore$ area $= \sqrt{(200 \times 11 \times 31 \times 158)} = 20\sqrt{(11 \times 31 \times 79)} = 20\sqrt{26939}$
$= 20 \times 164.13 = 3283$ *sq. yds*,
the value of which (there being 4840 *sq. yds* in an acre) will be $\frac{3283}{4840} \times$ £150 = £101 14*s* 11*d*.

9. Let C be the right angle: then, if A be the given angle $= 30°$, we have $B = 60°$, $a = c \sin A = 5\frac{1}{4} \times \frac{1}{2} = 2\frac{5}{8}$ *ft*, $b = c \sin B$ $= 5\frac{1}{4} \times \frac{1}{2}\sqrt{3} = 2\frac{5}{8} \times 1.732 = 4.5465$ *ft*, and area $= \frac{1}{2}ab = \frac{1}{2} \times 2\frac{5}{8} \times 4.5465$ $= 6$ *sq. ft* nearly.

0. Let the angles be $30°, \ 30° + x, \ 30° + 2x$; then their sum $= 90° + 3x = 180°$, or $x = 30°$, and the angles are $30°, 60°, 90°$; hence the triangle is right-angled, and
$a = 100$ *yds*, $b = 100 \tan 60° = 100\sqrt{3} = 173.20$ *yds;*
and area $= \frac{1}{2}ab = 8660$ *sq. yds* = 1A. 3R. 6P. $8\frac{1}{2}$ *sq. yds.*

1. $\sin C = \frac{c}{a} \sin A = \frac{20}{10} \times \frac{1}{2} = 1; \quad \therefore C = 90°$, and $B = 60°$; also
$b = c \cos 30° = 10\sqrt{3} = 10 \times 1.7320 = 17.$

2. Here a, b, c, are as $1, \frac{4}{3}, \frac{5}{3}$, and, the greatest angle being C, opposite to the greater side, $\cos C$ (106) $= \frac{1^2 + (\frac{4}{3})^2 - (\frac{5}{3})^2}{2 \times 1 \times \frac{4}{3}} = 0$;
hence $C = 90°$, and $\sin A = \frac{a}{c} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$, $\sin B = \frac{b}{c} = \frac{4}{3} \div \frac{5}{3} = \frac{4}{5}$.

13. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = -\dfrac{bc}{2bc} = -\tfrac{1}{2} = \cos 120^\circ.$

14. (1) $\sin\tfrac{1}{2}C = \sqrt{\dfrac{(s-a)(s-b)}{ab}} = \dfrac{1}{\sqrt{2}}$, since $\tfrac{1}{2}C = 45^\circ$;

$\therefore\ \tfrac{1}{2}ab = (s-a)(s-b) = \tfrac{1}{4}(b+c-a)(a+c-b) = \tfrac{1}{4}\{c^2 - (a-b)^2\};$

hence $c^2 = (a-b)^2 + 2ab = a^2 + b^2.$

(2) $\cos\tfrac{1}{2}C = \sqrt{\dfrac{s(s-c)}{ab}} = \dfrac{1}{\sqrt{2}}$;

$\therefore\ \tfrac{1}{2}ab = s(s-c) = \tfrac{1}{4}(a+b+c)(a+b-c) = \tfrac{1}{4}\{(a+b)^2 - c^2\};$

hence $c^2 = (a+b)^2 - 2ab = a^2 + b^2.$

(3) $\tan\tfrac{1}{2}C = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}} = 1$, since $\tan 45^\circ = 1$;

$\therefore\ (s-a)(s-b) = s(s-c)$, or $\tfrac{1}{4}\{c^2 - (a-b)^2\} = \tfrac{1}{4}\{(a+b)^2 - c^2\};$

hence $2c^2 = (a+b)^2 + (a-b)^2 = 2(a^2 + b^2).$

Ex. 21.

1. $R = \dfrac{a}{2\sin A} = \dfrac{a}{2\sin 60^\circ} = \dfrac{a}{\sqrt{3}};\ r = \dfrac{\text{area}}{s} = \dfrac{\tfrac{1}{2}a \times a \times \sin 60^\circ}{\tfrac{1}{2}(a+a+a)} = \dfrac{a}{2\sqrt{3}}.$

2. $s = \tfrac{1}{2}(a+a+c) = \tfrac{1}{2}(2a+c),\ s-a = \tfrac{1}{2}c = s-b,\ s-c = \tfrac{1}{2}(2a-c);$

hence $R\ (114) = \dfrac{a^2c}{4\sqrt{\{\tfrac{1}{16}(4a^2-c^2)c^2\}}} = \dfrac{a^2}{\sqrt{(4a^2-c^2)}};$

$r\ (115) = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\dfrac{\tfrac{1}{8}(2a-c)c^2}{\tfrac{1}{2}(2a+c)}} = \dfrac{c}{2}\sqrt{\dfrac{2a-c}{2a+c}}.$

3. By (115 N.B.) we have $AF \,.\, BD \,.\, CE = AE \,.\, BF \,.\, CD$ $= (s-a)(s-b)(s-c) = sr^2\ (115) = \tfrac{1}{2}(a+b+c)r^2.$

4. By (116)

$$BD^2 = a^2 + d^2 - 2ad\cos A = (a^2+d^2) - \frac{ad}{ad+bc}\{(a^2+d^2) - (b^2+c^2)\}$$

$$= \frac{bc(a^2+d^2) + ad(b^2+c^2)}{ad+bc} = \frac{(ac+bd)(ab+cd)}{ad+bc}.$$

We may proceed in the same way to find the value of AC^2: or we may from the above write down at once, by Symmetry, the value of AC^2, observing that, in the expression for BD^2, the terms ad, bc, in the den^r are the products of the sides which contain the angles A and C, respectively, and therefore, in the den^r of the expression for AC^2, the terms will be the products of the sides which contain the angles B and D, that is, the den^r will be $ab + cd$, and therefore

$$BD^2 = \frac{ab(c^2+d^2) + cd(a^2+b^2)}{ab+cd} = \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

Hence we have $AC.BD = ac + bd = AB.CD + BC.AD$; and $AC : BD :: ab+cd : ad+bc :: AB.BC+CD.DA : BA.AD+BC.CD$.

5. If the figure be drawn, it will be seen that (to radius unity)

$$AC = \text{chord}(2A + 2B) = 2\sin(A+B),\quad BD = 2,$$
$$AB = \text{chord}\,2A = 2\sin A,$$
$$CD = \text{chord}(180^\circ - BC) = 2\sin(90^\circ - B) = 2\cos B,$$
$$BC = \text{chord}\,2B = 2\sin B,$$
$$AD = \text{chord}(180^\circ - AB) = 2\sin(90^\circ - A) = 2\cos A;$$

∴ since, by the *datum* in the preceding example, we have $AC.BD = AB.CD + AD.BC$, it follows that

$$2\sin(A+B)\times 2 = 2\sin A \times 2\cos B + 2\cos A \times 2\sin B, \text{ \&c.}$$

6. Area $ABD = \frac{1}{2}$ base × perpendicular $= \frac{1}{2}BD \times AE \sin E$;

$$\text{so area } CBD = \tfrac{1}{2}BD \times CE \sin E;$$

$$\therefore \text{area } ABCD = \tfrac{1}{2}BD(AE + CE)\sin E = \tfrac{1}{2}BD.AC\sin E.$$

Now, if $ABCD$ be inscribible in a circle, as in (116), we have its area $= \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$, which, by the above, $= \frac{1}{2}AC.BD\sin E = \frac{1}{2}(ac+bd)\sin E$ by the foregoing Ex. 4, or by *Euc.* VI. D.

7. By (115) area $S = rs = r_1(s-a) = r_2(s-b) = r_3(s-c)$;

$$\therefore \frac{r}{r_1} + \frac{r}{r_2} + \frac{r}{r_3} = \frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} = \frac{3s-(a+b+c)}{s} = \frac{3s-2s}{s} = 1;$$

$$\text{hence } \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}, \text{ or } r = \text{\&c.}$$

8. $R = \dfrac{a}{2\sin A}$, and area $S = rs = r_1(s-a)$.

$$\text{Hence (1) } \frac{r}{R} = \frac{2S\sin A}{as} = \frac{2(\frac{1}{2}ab\sin C)\sin A}{a\times\frac{1}{2}(a+b+c)} = \frac{2b\sin C\sin A}{a+b+c}$$

$$= (110)\ \frac{2\sin A\sin B\sin C}{\sin A + \sin B + \sin C} = \frac{2\sin A\sin B\sin C}{4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}\ (15.\ 4)$$

$$= 4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

$$\text{So (2) } \frac{r_1}{R} = \frac{2S\sin A}{a(s-a)} = \text{(as above) } \frac{2\sin A\sin B\sin C}{\sin B + \sin C - \sin A}$$

$$= \frac{2\sin A\sin B\sin C}{4\cos\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C} = 4\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C,$$

ince $\sin B + \sin C - \sin A = 2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C) - 2\sin\frac{1}{2}A\cos\frac{1}{2}A$

$$= 2\cos\tfrac{1}{2}A\,\{\cos\tfrac{1}{2}(B-C) - \cos\tfrac{1}{2}(B+C)\},$$

or $\sin\frac{1}{2}(B+C) = \sin(90^\circ - \frac{1}{2}A) = \cos\frac{1}{2}A$, and $\sin\frac{1}{2}A = \cos\frac{1}{2}(B+C)$,)

$$= 2\cos\tfrac{1}{2}A\,(2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C).$$

Hence also (3) $\dfrac{r}{r_1} = \tan\frac{1}{2}B\tan\frac{1}{2}C$.

9. (1) $R = \dfrac{a}{2\sin A} = \dfrac{1}{2}\sqrt[3]{\left(\dfrac{a^3}{\sin^3 A}\right)} = \dfrac{1}{2}\sqrt[3]{\left(\dfrac{a}{\sin A}\cdot\dfrac{b}{\sin B}\cdot\dfrac{c}{\sin C}\right)}$:

(2) $R = \dfrac{a}{2\sin A}$, $r = \dfrac{S}{s} = \dfrac{\frac{1}{2}bc\sin A}{s}$; $\therefore\ 2Rr = \dfrac{abc}{2s}$.

10. The hypothenuse must be the diameter of the circumscribing circle, so that $R = \frac{1}{2}c$; and $r = \dfrac{S}{s} = \dfrac{\frac{1}{2}ab}{s} = \dfrac{ab}{a+b+c}$; hence

$$R + r = \tfrac{1}{2}c + \frac{ab}{a+b+c} = \frac{(a+b)c + c^2 + 2ab}{2(a+b+c)} = \frac{(a+b)c + a^2 + b^2 + 2ab}{2(a+b+c)}$$
$$= \frac{(a+b)c + (a+b)^2}{2(a+b+c)} = \frac{(a+b)(c+a+b)}{2(a+b+c)} = \tfrac{1}{2}(a+b).$$

11. From P draw PD perpendicular on AB; then we have $\angle APD = \frac{1}{2}\angle APB$ (at the centre) $= \angle ACB$ (at the circumference) $= C$, and $\therefore\ \angle PAB = 90^\circ - C = \frac{1}{2}(A + B + C) - C = \frac{1}{2}(A + B - C)$; and $\angle QAB = \frac{1}{2}A$; $\therefore\ \angle PAQ = \angle PAB - \angle QAB$ or $\angle QAB - \angle PAB$ $= \frac{1}{2}(A + B - C) \sim \frac{1}{2}A = \frac{1}{2}(B \sim C)$; and so with the others.

12. Let $ABCD$ be the quadrilateral figure, so that $AB = BC = 3$, and $CD = DA = 4$: it is plainly bisected by the diagonal BD, and therefore, since it may be inscribed in a circle, BD must be the diameter of that circle, and the angles BAD, CAD, must be right angles.

Hence area $ABCD = 2 \times$ area $BAD = 2 \times \frac{1}{2}AB \,.\, AD = 12$; radius of circumscribing circle $= \frac{1}{2}BD = \frac{1}{2}\sqrt{(AB^2 + AD^2)} = 2\frac{1}{2}$; and, if O be the centre of inscribed circle,

area $ABCD$ = area OAB + area OBC + area OCD + area ODA,

or $12 = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr + \frac{1}{2}dr = \frac{1}{2}(a+b+c+d)r = 7r$; $\therefore\ r = 1\frac{5}{7}$.

Ex. 22.

(1) Given a, A: then $B = 90^\circ - A = 62^\circ\ 14'\ 29''$:

$\log b = 10 + \log a - \text{L}\tan A = 12.0915147 - 9.7212462$
$= 2.3702685 = \log 234.5678$;

$\log c = 10 + \log a - \text{L}\sin A = 12.0915147 - 9.6681505$
$= 2.4233642 = \log 265.0721$.

(2) Given c, A: then $B = 90^\circ - A = 62^\circ\ 14'\ 29''$:

$\log a = \log c + \text{L}\sin A - 10 = 2.4233642 + 9.6681505 - 10$
$= 2.0915147 = \log 123.4567$:

$\log b = \log c + \text{L}\cos A - 10 = 2.4233642 + 9.9469043 - 10$
$= 2.3702685 = \log 234.5678$.

3) Given a, b: then $\mathrm{L}\tan A = 10 + \log a - \log b$
$= 12.0915147 - 2.3702685 = 9.7212462 = \mathrm{L}\tan 27^\circ\, 45'\, 31''$:
$\therefore\ A = 27^\circ\, 45'\, 31''$, $B = 90^\circ - A = 62^\circ\, 14'\, 29''$,
and $\log c = 10 + \log a - \mathrm{L}\sin A = \log 265.0721$, as above in (1).

4) Given a, c: then $\mathrm{L}\sin A = 10 + \log a - \log c$
$= 12.0915147 - 2.4233642 = 9.6681505 = \mathrm{L}\sin 27^\circ\, 45'\, 31''$;
$\therefore\ A = 27^\circ\, 45'\, 31''$, $B = 90^\circ - A$,
and $\log b = \log c + \mathrm{L}\cos A - 10$, as in (2).

Or we may find b thus:
og$b = \frac{1}{2}\{\log(c+a) + \log(c-a)\} = \frac{1}{2}\{2.5894249 + 2.1511121\} = 2.3702685.$

Ex. 23.

1) Given a, A, B: then $C = 180^\circ - A - B = 70^\circ$:

$$\log b = \log a + \mathrm{L}\sin B - \mathrm{L}\sin A = \begin{array}{r} 2.8839067 \\ +\ 9.9607302 \\ \hline 12.8446369 \\ -\ 9.8417713 \\ \hline 3.0028656 \end{array} = \log 1006.62:$$

$$\log c = \log a + \mathrm{L}\sin C - \mathrm{L}\sin A = \begin{array}{r} 2.8839067 \\ +\ 9.9729858 \\ \hline 12.8568925 \\ -\ 9.8417713 \\ \hline 3.0151212 \end{array} = \log 1035.43.$$

2) Given a, b, A:

$$\mathrm{L}\sin B = \mathrm{L}\sin A + \log b - \log a = \begin{array}{r} 9.8417713 \\ +\ 3.0028656 \\ \hline 12.8446369 \\ -\ 2.8839067 \\ \hline 9.9607302 \end{array} = \mathrm{L}\sin 66^\circ \text{ or } \mathrm{L}\sin 114^\circ,$$

either of which is possible, since $b > a$.

Hence $C = 180^\circ - A - B = 70^\circ$ or 22°:

and $\log c = \log a + \mathrm{L}\sin C - \mathrm{L}\sin A =$ (if $C = 70^\circ$) $\log 1035.43$, as in (1);

$$\text{or} = (\text{if } C = 22^\circ)\ \begin{array}{r} 2.8839067 \\ +\ 9.5735754 \\ \hline 12.4574821 \\ -\ 9.8417713 \\ \hline 2.6157108 \end{array} = \log 412.7725.$$

Ex. 24.

(3) Given a, b, C: then $A + B = 180° - C = 110°$:

and, (b being here (122) the greatest of the two given sid

$\text{L} \tan \frac{1}{2}(B - A) = 20 + \log(b - a) - \text{L} \tan 35° - \log(b + a)$

$= 22.3823555 - 13.0937033 = 9.2886522 = \text{L} \tan$

hence $B + A = 110°$, $B - A = 22°$; $\therefore B = 66°$, $A = 44°$,

may now be found as in Ex. 23 (1).

(4) Given a, b, c: then, since $\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, we ha

$\text{L} \tan \frac{1}{2}A = 10 + \frac{1}{2}\{\log(s-b) + \log(s-c) - \log s - \log(s-a)\}$

$= 10 + \frac{1}{2}\{5.1651992 - 5.9523799\} = 10 - .3935903$

$= 9.6064096 = L \tan 22°$; $\therefore A = 44°$:

$\text{L} \tan \frac{1}{2}C = 10 + \frac{1}{2}\{\log(s-a) + \log(s-b) - \log s - \log(s -$

$= 10 + \frac{1}{2}\{5.4040164 - 5.7135627\} = 10 - .15477$

$= 9.8452268 = \text{L} \tan 35°$; $\therefore C = 70°$:

and $B = 180° - A - C = 66°$.

Ex. 25.

1. Let the offsets, on opposite sides of AC, be EB (=
and FD (= 420); then (127 (1)) area $= \frac{1}{2}AC\,(BE +$
$= \frac{1}{2} \times 750 \times 700 = 262500$ *square links* = 2.625 *acres* = 2A. 2R
2. $BC = AC \tan BAC$: $\therefore \log BC = \log 200 + \text{L} \tan 25° 10$
$= 2.3010300 + 9.6719628 - 10 = 1.9729928 = \log 93.97$:
$\therefore BC = 93.97 = 94$ *feet*, nearly.
3. A, C, the observers, B the balloon: $AC = 1000$ *yds*, $\angle A$
$\angle C = 54°$, and, $\therefore \angle B = 90°$; hence, drawing BD perp. to
$BD = AB \sin A = (AC \cos A) \sin A = \frac{1}{2}AC \sin 2A = 500$
$= 125\sqrt{(10 + 2\sqrt{5})}$ by (100) $= 125\sqrt{14.472134} = 125 \times$
$= 475\frac{1}{2}$ *yds*.
4. Let BC be the tower; A, D, the two places of observ
then $AD = 100$ *yds*, $\angle BAC = 60°$, $\angle BDC = 30°$; hence $\angle$
$= \angle(BAC - BDC) = 30°$, and $\therefore AB = AD = 100$ *yds*;
$= AB \sin 60° = 50\sqrt{3} = 50 \times 1.732 = 86.6$ *yds* = 260 *ft*, ne
5. Let the offsets (both on the same side of AB) be EB (
and $FC (= 435)$; then (127 (3)) area $= \frac{1}{2}(AF \times BE + DE$
$= \frac{1}{2}(710 \times 147 + 820 \times 435) = 230535$ *square links* = 2.3053
= 2A. 1R. 9P., nearly.
6. Given $AB = AC = 200$ *yds*, $\angle ABC = \angle ACB = 36°$;
then (drawing AD perp. on BC) $BC = 2BD = 2AB$ c
$= 100(\sqrt{5} + 1)$ by (100) $= 100 \times 3.2360 = 324$ yds, nearl

7. Let A be the ship's bottom, B the top of the rock; then $BC = 500$ *ft*, $\angle BAC = 18°$; and $\therefore$ $AC = BC \cot 18°$, and $\log AC = \log 500 + \text{L}\cot 18° - 10 = \log 500 + (20 - \text{L}\tan 18°) - 10$ by (155) $= 2.6989700 + 10 - 9.5117760 = 3.1871940 = \log 1538.8$; hence $AC = 1539$ *ft*, nearly.

8. A the top of the tower, $AB = 108$ *ft*; CD the column, CE a horizontal line, cutting AB in E; $\angle ACE = 30°$, $\angle ADB = 60°$: then $AD = AB \times \operatorname{cosec} ADB = 108 \times \frac{2}{3}\sqrt{3} = 72\sqrt{3}$; then, in triangle ACD, we have $AD = 72\sqrt{3}$, $\angle ADC = 30°$, $\angle ACD = 90° + 30° = 120°$, and $\therefore$ $\angle CAD = 30°$;

$$\therefore\ CD = AD\frac{\sin CAD}{\sin ACD} = 72\sqrt{3} \times \frac{\sin 30°}{\sin 120°} = 72\,ft.$$

9. Given $AC = 75$ *yds*, $CD = 60$ *yds*: then $\sin A = \frac{60}{75} = \frac{4}{5}$, and $\cos A = \sqrt{(1 - \sin^2 A)} = \frac{3}{5}$: hence $BC = AC \tan A = 75 \times \frac{4}{3} = 100$ *yds*, $AB = AC \sec A = 75 \times \frac{5}{3} = 125$ *yds*, area $= \frac{1}{2}AC \times BC = 3750$ *s.y.*

10. $AB = 1000$ *yds*, $\angle C = 180° - A - B = 64° 40'$;

$$\therefore\ AC = AB\frac{\sin 83°\ 10'}{\sin 64°\ 40'} = 1000\frac{\cos 6°\ 50'}{\cos 25°\ 20'},\ \text{and}$$

$\log AC = 3 + \text{L}\cos 6°\ 50' - \text{L}\cos 25°\ 20' = 3.0408154 = \log 1098.5$; $\therefore$ $AC = 1098.5 = 1099$ *yds*, nearly.

11. Given $\angle A = 60°$, $CD = 100$ *yds*: then, since $DA = DB$, and $\angle ACB = 90°$, D must be the centre of the circumscribing circle; hence $AD = DC = DB$, and $AB = 2CD = 200$ *yds*, $AC = AB \cos 60° = 100$ *yds*, $BC = AB \sin 60° = 100\sqrt{3} = 173$ *yds*, nearly; and area $= \frac{1}{2}AC \times BC = 8650$ *sq. yds.*

12. AB the tower, CD the pillar, CE horizontal, cutting AB in E: then $CD = 100$ *ft*, $\angle ACE = 30°$, $\angle ADB = 60°$; and, in triangle ACD, we have $CD = 100$ *ft*, $\angle ADC = 30°$, $\angle ACD = 90° + 30° = 120°$, and $\therefore$ $\angle CAD = 30°$; hence $AC = CD = 100$, and $AE = AC \sin 30° = 50$, and $\therefore$ $AB = AE + CD = 150$ *ft*.

13. $AB = BC\dfrac{\sin 30°}{\sin 54°} = BC\dfrac{2}{\sqrt{5}+1} = BC\dfrac{\sqrt{5}-1}{2} = 150\,(1.2360) = 185$ *yds*.

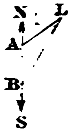
14. DE the object, A, B, C, the three successive places of observation; $AB = 80$ *ft*, $BC = 30$ *ft*, ($\therefore$ $AC = 50$ *ft*,) $\angle DAE = A$, $\angle DBE = 90° - A$, $\angle DCE = 2A$: hence $\angle ADB = \angle(DBE - DAE) = 90° - 2A$, and $\angle ADC = \angle(DAB - DCB) = A$.

$$\text{Now } AD = AB\,\frac{\sin DBA}{\sin ADB} = 80 \times \frac{\cos A}{\cos 2A};$$

and, again, $AD = AC\,\dfrac{\sin DCA}{\sin ADC} = 50 \times \dfrac{\sin 2A}{\sin A} = 100 \cos A$:

$\therefore\ 80\,\dfrac{\cos A}{\cos 2A} = 100 \cos A$, or $\cos 2A = \frac{4}{5}$; and $DE = AD \sin A$ $= (100 \cos A) \sin A = 50 \sin 2A = 50 \times \frac{3}{5} = 30$ *ft.*

15. Let A, B be the two positions of the ship, L the lighthouse; then $\angle NAL = 45^\circ$, $\angle NBL = 30^\circ$, and therefore $\angle ALB = \angle(NAL - NBL) = 15^\circ$: hence



$$AL = AB\,\frac{\sin 30^\circ}{\sin 15^\circ} = 2AB \cos 15^\circ = 2AB \cos(45^\circ - 30^\circ)$$

$= 12 \times \dfrac{\sqrt{3}+1}{2\sqrt{2}} = 3 \times (\sqrt{6}+\sqrt{2}) = 3 \times (3.86) = 11.6$ *miles*, nearly:

$BL = AB\,\dfrac{\sin 45^\circ}{\sin 15^\circ} = 6 \times \dfrac{2}{\sqrt{3}-1} = 6 \times \dfrac{\sqrt{3}+1}{2} = 3 \times (2.73) = 8.2$ *miles.*

16. By (127 (2)) area $= \frac{1}{2}(230 + 480)\,200 = 71000$ *square yards* $= 14$A. 2R. 27P. $3\frac{1}{4}$ *sq. yds.*

17. Suppose each road stretched out into a straight line: then the vertical *height* of the hill will be the perpendicular of a right-angled triangle, in which the *length* of the hill will be the hypothenuse: hence, in each case, we have height = length × sine of inclination $= 1\frac{2}{3} \times \frac{1}{5}$ (in one case) $= l \times \frac{1}{12}$ (in the other,) if l be the length required: $\therefore l = 12 \times 1\frac{2}{3} \times \frac{1}{5} = 4$ *miles.*

18. By (127 (1)) area $= \frac{1}{2} \times 108\frac{1}{2} \times (56\frac{1}{4} + 60\frac{3}{4}) = 6347\frac{1}{4}$ *sq. ft* $= 23$P. $9\frac{1}{4}$ *sq. yds* $2\frac{1}{4}$ *sq. ft* $= 23$P. 9 *sq. yds* $4\frac{1}{2}$ *sq. ft.*

19. BC the tower, A, D, the two positions of the observer;

$$AD = a,\ \angle BAC = a,\ \angle BDC = \tfrac{1}{2}a;$$

hence $\angle ABD = \angle(BAC - BDC) = \frac{1}{2}a = \angle BDA$, $AB = AD = a$;

$$\therefore\ BC = AB \sin BAC = a \sin a,\ AC = a \cos a.$$

20. Given $AC = 80$ *ft*, $CD = 40$ *ft*, $\angle ACB = 144^\circ$, $\angle CDB = 36^\circ$ $= \angle BCD$; $\therefore BC = BD$, and $CD = 40 = 2CE$, (if BE be drawn perp. on CD,) $= 2BC \cos 36^\circ$, or $BC = \dfrac{20}{\cos 36^\circ} = \dfrac{80}{\sqrt{5}+1} = 20(\sqrt{5}-1)$.

Now we have $AB^2 = AC^2 + BC^2 - 2AC.BC \cos 144^\circ$

$$= 80 + 20^2(6 - 2\sqrt{5}) + 80 \times 40,$$

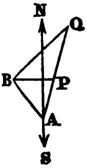
(since, from above, $40 = 2BC \cos 36^\circ = -\,2BC \cos 144^\circ$,)

$$= 20^2(16 + 6 - 2\sqrt{5} + 8) = 20^2(25.5280);$$

$$\therefore\ AB = 20 \times 5.05 = 101 \text{ ft.}$$

21. B the top of the hill, A, D, the two objects: $AD = \frac{1}{10}$ *mile*, $\angle BAC = 45°$, $\angle BDC = 30°$: then, in triangle ABD, we have $AB = \frac{1}{10}$ *mile* $= 176$ *yds*, $\angle BDA = 30°$, $\angle BAD = 135°$, and $\therefore \angle ABD = 15°$; hence $BD = AD\,\frac{\sin 135°}{\sin 15°} = 176 \times \frac{2}{\sqrt{3}-1}$ $= 176 \times (\sqrt{3}+1) = 176 \times (2.7320) = 480.8$ *yds* $= 1442$ *ft*, nearly.

22. A, B, the two positions of the ship, $\angle NAP = 15°$, $\angle NAB = 45°$, $\angle QBP = 45°$: hence $\angle ABQ = 90°$, $\angle BAP = 60°$, and $AQ = AB \sec 60° = 2AB = 10$ *miles*,

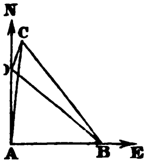


$$AP = AB\,\frac{\sin ABP}{\sin APB} = 5 \times \frac{\sin 45°}{\sin 75°} = 5 \times \frac{2}{\sqrt{3}+1}$$

$= 5(\sqrt{3}-1)\,m.$: $\therefore PQ = 10 - 5(\sqrt{3}-1) = 5(3-\sqrt{3}) = 63\,m.$

23. In the triangle BCD, we have $CD = 100$ *yds*, $\angle BDC = 36°$, $\angle BCD = 126°$, and $\therefore \angle CBD = 18°$; hence $BD = CD\,\frac{\sin 126°}{\sin 18°}$ $= 100 \times \frac{\sin 54°}{\sin 18°} = 100 \times \frac{\sqrt{5}+1}{\sqrt{5}-1} = 100 \times \frac{(\sqrt{5}+1)^2}{4} = 50(3+\sqrt{5})$ *yds*: $\therefore AB^2 = AD^2 + BD^2 - 2AD.BD \cos 36° = 400^2 + 2500(14+6\sqrt{5})$ $- 40000\,(3+\sqrt{5})\,\frac{\sqrt{5}+1}{4} = 115000 - 25000\sqrt{5} = 59098$ *sq. yds*: $\therefore AB = 243$ *yds*, nearly.

4. CD the tower, perp. to plane of paper, A, B, the two positions of the observer, $\angle CAD = 60°$, $AB = 400$ *ft*; then $AD = CD \cot 60° = \frac{100}{3}\sqrt{3}$, $BD^2 = AB^2 + AD^2$ $= 100^2(16 + \frac{1}{3}) = \frac{400000}{3}$; $\therefore BD = \frac{700}{3}\sqrt{3}$, and $\tan CBD = \frac{CD}{BD} = \frac{1}{7}\sqrt{3} = .2474358 = \tan 13°\,53'\,52''$, (since $D = 3087$, $d = 2695$, and $\therefore n = \frac{2695}{3087} \times 60 = 52$).



Ex. 26.

1. $\log 8 = 3\log 2 =$ &c; $\log 9 = 2\log 3 =$ &c; $\log 12 = \log 3 + \log 4$ $= \log 3 + 2\log 2 =$ &c.

2. $\log 20 = \log 10 + \log 2 = 1 + \log 2$; $\log 25 = 2\log 5 = 2(1 - \log 2)$; $\log 60 = \log 10 + \log 6 = 1 + \log 2 + \log 3$.

3. $\log \frac{1}{3} = -\log 3 = -.4771213 = -1 + .5228787 =$ &c; $\log \frac{1}{4} = -2\log 2$ $= -.60206 = -1 + .39794 =$ &c; $\log \frac{2}{5} = \log \frac{4}{10} = 2\log 2 - 1$ $= -1 + .60206 =$ &c.

4. $\log\frac{3}{100} = \log 3 - 2 = -2 + .4771213 = \&c$; $\log\frac{1}{30} = -\log 3 - \log 10 = -.4771213 - 1 = -2 + .5228787 = \&c$; $\log\frac{33}{10000} = \log 3 + \log 11 - 4 = \&c$.

5. $\log\frac{18}{10} = \log 2 + 2\log 3 - 1 = \&c$; $\log 140 = \log 2 + \log 7 + 1 = \&c$; $\log\frac{144}{100} = \log 9 + \log 16 - 2 = 2\log 3 + 4\log 2 - 2 = \&c$.

6. $\log\frac{625}{10000} = 4\log 5 - 4 = 4(1 - \log 2) - 4 = -4\log 2 = -1.20412 = -2 + .79588 = \&c$; $\log\frac{1}{1210} = -2\log 11 - 1 = -2.0827854 - 1 = -4 + .9172146 = \&c$; $\log\frac{105}{10} = \log 3 + \log 7 + \log 5 - 1 = \log 3 + \log 7 + (1 - \log 2) - 1 = \&c$.

7. $\log\frac{106}{10} = \log 53 - 1$; $\log\frac{21}{5} = \log\frac{42}{10} = \log 2 + \log 3 + \log 7 - 1$; $\log\frac{22}{5} = \log\frac{44}{10} = 2\log 2 + \log 11 - 1$.

8. $\log\frac{111}{10000} = \log 3 + \log 37 - 4$; $\log\sqrt{\frac{3}{2}} = \frac{1}{2}(\log 3 - \log 2)$; $\log\sqrt[3]{\frac{4}{3}} = \frac{1}{3}(2\log 2 - \log 3)$.

9. $\log\sqrt{\frac{1}{10}} = \frac{1}{2}\log\frac{1}{10} = \frac{1}{2}(-1) = -.5 = -1 + .5 = \&c$; $\log\sqrt[3]{\frac{2}{100}} = \frac{1}{3}(\log 2 - 2) = \frac{1}{3}(.30103 - 2) = \frac{1}{3}(-3 + 1.30103) = -1 + .4336766 = \&c$; $\frac{2}{3}\log\frac{12}{10} = \frac{2}{3}(2\log 2 + \log 3 - 1)$.

10. $3\log\frac{1}{7} = 3\,\mathrm{colog}\,7 = 3 \times \bar{1}.1549020 = \&c$; $\log(\frac{3}{4}\sqrt{\frac{11}{10}}) = \log 3 - 2\log 2 + \frac{1}{2}(\log 11 - 1) = \&c$; $\frac{2}{5}\log\frac{69}{1000} = \frac{2}{5}(\log 3 + \log 23 - 3)$.

11. $\frac{1}{4}\,\mathrm{colog}\,2 + \frac{1}{3}(\log 2 + \mathrm{colog}\,3) + \frac{1}{2}(\log 3 + 2\,\mathrm{colog}\,2) = \frac{1}{4}(\bar{1}.6989700) + \frac{1}{3}(.3010300 + \bar{1}.5228787) + \frac{1}{2}(.4771213 + \bar{1}.3979400) =$ (see Ex. 3.) $\frac{1}{4}(\bar{1}.6989700) + \frac{1}{3}(\bar{1}.8239087) + \frac{1}{2}(\bar{1}.8750613) =$ (see Ex. 3) $\bar{1}.9247425 + \bar{1}.9413029 + \bar{1}.9375306 = \bar{1}.8035760$.

12. $\frac{1}{2}[\mathrm{colog}\,2 + \frac{1}{3}\{\log 2 + \mathrm{colog}\,3 + \frac{1}{4}(\log 3 + 2\,\mathrm{colog}\,2)\}]$
$= \frac{1}{2}[\bar{1}.6989700 + \frac{1}{3}\{\bar{1}.8239087 + \frac{1}{4}(\bar{1}.8750613)\}]$ as in (11)
$= \frac{1}{2}\{\bar{1}.6989700 + \frac{1}{3}(\bar{1}.8239087 + \bar{1}.9687653)\}$
$= \frac{1}{2}\{\bar{1}.6989700 + \frac{1}{3}(\bar{1}.7926740)\} = \frac{1}{2}\{\bar{1}.6989700 + \bar{1}.9308913\}$
$= \frac{1}{2}(\bar{1}.6298613) = \bar{1}.8149306$.

Ex. 27.

3. $\log 20000 = 4.3010300$, $\log 20001 = 4.3010517$; $\therefore D = 117$, &c.

4. $\log 31000 = 4.4913617$, $\log 31001 = 4.4913757$; $\therefore D = 140$, and (i) for 31000.23, $\delta = \frac{23}{100} \times 140 = 23 \times 1.4 = 32$; (ii) for $31000\frac{1}{4}$, $\delta = \frac{1}{4} \times 140 = 35$; (iii) $31\frac{3}{4000} = 31.00075$, and for 31000.75, or $31000\frac{3}{4}$, $\delta = \frac{3}{4} \times 140 = 105$: in each case the value of δ is to be added to the *mantissa* for 31, and the Characteristic determined by inspection.

5. $\log\sqrt{2\frac{1}{3}} = \frac{1}{2}\log\frac{7}{3} = \frac{1}{2}(\log 7 - \log 3) = .1839883$;
$\log\sqrt[3]{3\frac{1}{4}} = \frac{1}{3}\log\frac{13}{4} = \frac{1}{3}(\log 13 - 2\log 2) = .1706278$;
$\log\sqrt[4]{4\frac{1}{5}} = \frac{1}{4}\log\frac{21}{5} = \frac{1}{4}\log\frac{42}{10} = \frac{1}{4}(\log 2 + \log 3 + \log 7 - 1) = .1558123$;
$\log\sqrt[5]{5\frac{1}{6}} = \frac{1}{5}\log\frac{31}{6} = \frac{1}{5}(\log 31 - \log 2 - \log 3) = .1426420$

.8530704,

from which, taking the lowest of the two given mantissæ, we have $\delta = 27$; also $D =$ diff. of given mantissæ $= 97$, for which we may form a Table of Pro. Parts, or, at once, by (143),

$$d = \frac{\delta}{D} = \frac{27}{97} = .27.$$

6. $\frac{1}{2}\log 357.1328 - \frac{1}{3}\log 35712.75 = \frac{1}{2}(2.5528297) - \frac{1}{3}(4.5528233)$
$= 1.2764148 - 1.5176077 = -1 + .7588071 = \bar{1}.7588071$:
now, for the given mantissæ, $D = 75$, and, taking the lowest of them from that just found, $\delta = 11$; $\therefore d = \frac{11}{75} = .14$; and the number required is, therefore, .5738614, the decimal point being set before the first figure, since the Characteristic is $\bar{1}$.

Ex. 28.

1. (i) D (for $60''$) $= 6261$; $\therefore d$ for $25'' = \frac{25}{60} \times 6261 = 2609$,
and $\text{L}\sin 11^\circ\ 24'\ 25'' = 9.2961738$:
(ii) $d = 746$; $\therefore n = \frac{746}{6261} \times 60 = 7$, and $A' = 11^\circ\ 24'\ 7''$.

2. $\text{L}\cos 60^\circ = 10 + \text{log-cos}\,60^\circ = 10 + \log\frac{1}{2} = 10 + \text{colog}\,2 = 9.6989700$:
(i) D (for $60''$)$= -2189$; $\therefore d$ for $20'' = \frac{20}{60} \times -2189 = -729.6 = -730$,
and $\text{L}\cos 60^\circ\ 0'\ 20'' = 9.6988970$:
(ii) $\text{L}\sin 29^\circ\ 59'\ 15'' = \text{L}\cos 60^\circ\ 0'\ 45''$;
$d = \frac{45}{60} \times -2189 = -1642$, and $\text{L}\sin 29^\circ\ 59'\ 15'' = 9.6988058$.

3. $\text{L}\tan 45^\circ = 10 + \text{log-tan}\,45^\circ = 10 + \log 1 = 10$; $\therefore D$ (for $60''$) $= 2527$: and $\text{L}\cot 44^\circ\ 59'\ 20'' =$ by (155) $20 - \text{L}\tan 44^\circ\ 59'\ 20''$, for which $d = \frac{20}{60} \times 2527 = 842$, and $\therefore \text{L}\tan 44^\circ\ 59'\ 20'' = 9.9998315$, and $\text{L}\cot 44^\circ\ 59'\ 20'' = 10.0001685$.

4. $\text{L}\,\text{cosec}\,30^\circ = 10 + \text{log-cosec}\,30^\circ = 10 + \log 2 = 10.3010300$; and $\therefore D$ (for $60''$) $= -2187$:
(i) $\text{L}\sin 30^\circ\ 0'\ 45'' = 20 - \text{L}\,\text{cosec}\,30^\circ\ 0'\ 45''$,
for which $d = \frac{45}{60} \times -2187 = -1640$, and $\therefore \text{L}\,\text{cosec}\,30^\circ\ 0'\ 45''$
$= 10.3008660$, and $\text{L}\sin 30^\circ\ 0'\ 45'' = 9.6991340$:
(ii) $\text{Log-sin}^{-1}9.6990999 = (155)\ \text{Log-cosec}^{-1}(20 - 9.6990999)$
$= \text{Log-cosec}^{-1}10.3009001$, for which $d = -1299$;
$\therefore n = \frac{1299}{2187} \times 60 = 35.6 = 36$, and $A' = 30^\circ\ 0'\ 36''$.

5. $\sin 2A = 2\sin A\cos A$; $\therefore \text{log-sin}\,2A = \log 2 + \text{log-sin}\,A + \text{log-cos}\,A$,
or $(\text{L}\sin 2A - 10) = \log 2 + (\text{L}\sin A - 10) + (\text{L}\cos A - 10)$, &c.

$$\text{L}\sin 30^\circ = 10 + \log\tfrac{1}{2} = 10 + \text{colog}\,2 = 9.6989700;$$

$$\text{L}\cos 30^\circ = 10 + \log(\tfrac{1}{2}\sqrt{3}) = 10 + \text{colog}\,2 + \tfrac{1}{2}\log 3$$
$$= 9.6989700 + .2385606 = 9.9375306.$$

6. $L \text{ vers } A = 10 + \log \text{vers } A = 10 + \log (2 \sin^2 \frac{1}{2} A)$

$= 10 + \log 2 + 2 \log \sin \frac{1}{2} A = 10 + \log 2 + 2 (L \sin \frac{1}{2} A - 10) = \&c:$

$L \text{ vers } 60^\circ = \log 2 + 2 L \sin 30^\circ - 10 = .3010300 + 19.3979400 - 10$

(by the last Example) $= 9.6989700$:

but, more simply,

$L \text{ vers } 60^\circ = 10 + \log \text{vers } 60^\circ = 10 + \log \frac{1}{2} = 10 - \text{colog } 2 = 9.6989700.$

ERRATA IN PLANE TRIGONOMETRY, PART I.

PAGE

67, line 11, **for** $=0$ **read** $=-1$.

68, The words 'For all values of A, B, C, shew that' should *follow*, not *precede*. Ex. 18. 10.

73, line 6, **for** *distance* **read** *height*.

83, line 9, **for** $(1-\cos\delta)$ **read** $(1-\cos\delta)$.

– line 10, **for** $\cot\frac{1}{2}\delta$ **read** $\tan\frac{1}{2}\delta$.

– line 28, in num^r, **for** $-\sin\delta$ **read** $-\tan\delta$.

– line 33, in den^r, **for** $\tan\frac{1}{2}\delta$ **read** $\tan\delta$.

96, Ex. 21. 6, **for** S (four times) **read** s.

111, line 4, **for** 0 *and* $+\infty$ **read** $-\infty$ *and* 0.

– line 5, **for** 0 *and* $-\infty$ **read** 0 *and* $+\infty$.

ANSWERS.

Ex. 20. 5, should be 3A. 2R. 18P. 11¼ *sq. yds.*

Ex. 20. 10, should be 1A. 3R. 6P. 8½ *sq. yds.*

Ex. 28. 2, should be 9.6988970, 9.6988059.

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APPENDIX

&c.

The following remarks are necessary to adapt the Key to the *Second* Edition of the Text.

Ex. **1** and **2** are the same in both Editions, as well as Ex. **3**. 1-4, **4**. 1-6, **4**. 8.

The Solutions of Ex. **3**. 5, 6, **4**. 7, 9, 10, **5**. 7, &c., that is, of all the *new* examples, are given below.

Second Edition.		*First Edition and Key.*
6. 7, 8, 9, 10	correspond to	**3**. 9, 10, 11, 12.
5. 1, 2, 8, 9, 10		**4**. 7, 9, 10, 11, 12.
5. 3, 4, 5, 6		**3**. 7, 6, 8, 5.
10. 6, 7.		**9**. 8, 9.

Any other example of the Second Edition will be found at once in the Key, upon diminishing by unity the Egyptian numeral, which indicates the *set* in which it is found: thus Ex. **14**. 4 of the Second Edition will be Ex. **13**. 4 of the Key.

---

3. 5. $2\pi r = 25000$, or $2r = \dfrac{25000}{\pi} = \dfrac{2500}{18} \times \dfrac{180}{\pi} = \dfrac{1250}{9} \times 57.29577$ $= 7958$ *miles*.

3. 6. The circumference traced by its extremity $= 2\pi r = 14\pi$ *yds*; hence in $1'$ it describes $140\pi = 140 \times \frac{22}{7} = 440$ *yds* $= \frac{1}{4}$ *mile*, and in 1 *h* it describes 15 *miles*.

---

4. 7. $A = \frac{5}{6}\omega^\circ = \frac{5}{6} \times 57^\circ.29577 = 47^\circ\, 44'\, 47''$.

4. 9. By (27) $\theta = \dfrac{A}{200}\pi = \dfrac{1.0375}{200}\pi = .0051875 \times 3.14159 = .016297$ nearly; which angle in *degrees* $= \frac{9}{10} \times 1.0375 = .93375 = 56'\, 1\frac{1}{2}''$.

4. 10. $A = .7854 \times 57^\circ.29577 = 45^\circ$, nearly.

---

5. 7. Let $a =$ side of square, $r =$ radius of each circle $= \frac{1}{4}a$: then area required

$= a^2 - 4\pi r^2 = a^2 - \frac{1}{4}\pi a^2 = a^2(1 - .78539\&c) = .21460$ sq. ft. $= 31$ sq. in.

8. 10. In the figure of (56), draw *CN* perpendicular to *AB*, and produce *BA* &c. as in Ex. **7.** 12 of Key.

---

9. 6. Apply the rules of 60–63.

9. 7. $\cos(\theta - \frac{1}{2}\pi) = \cos(\frac{1}{2}\pi - \theta) = $ &c,
$\sin(\theta - \frac{3}{2}\pi) = -\sin(\frac{3}{2}\pi - \theta) = -(-\cos\theta) = $ &c,
$\tan(\theta - \frac{1}{2}\pi) = -\tan(\frac{1}{2}\pi - \theta) = $ &c.

---

10. 8. In the figure of (66), let *BC* be the tower above the level of the observer's head, and produce *CA* to *D*, so that $AD = 80$ *yds*; then $\angle BAC = 60°$, $\angle BDC$ or $\angle BDA = 30°$, and $\therefore \angle ABD = 30°$, (since *BAC* is the *exterior* angle of the triangle *ABD*,) and $AB = AD = 80$ *yds*: hence $BC = AB \sin 60° = 80 \times \frac{1}{2}\sqrt{3} = 40 \times 1.7320 = 69.280$ *yds* $= 208$ *ft*, nearly, and height of tower $= 208 + 6 = 214$ *ft.*

10. 9. $AB^2 = 108^2 + 144^2 = 36^2(9+16)$; $\therefore AB = 36 \times 5 = 180$ *yds*: and $CD.AB = 2$ area $ABC = AC.BC$,

whence $CD = 108 \times 144 \div 180 = 86.4$ *yds*;

also $\dfrac{BD}{BC} = \cos B = \dfrac{BC}{AB}$, or $BD = \dfrac{BC^2}{AB} = \dfrac{144 \times 144}{180} = 115.2$ *yds*;

and $AD = 180 - 115.2 = 64.8$ *yds.*

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13. 10. $$u = \frac{2\sin\frac{1}{2}(\frac{2}{3}A + \frac{4}{3}A)\cos\frac{1}{2}(\frac{2}{3}A - \frac{4}{3}A)}{2\cos\frac{1}{2}(\frac{2}{3}A + \frac{4}{3}A)\cos\frac{1}{2}(\frac{2}{3}A - \frac{4}{3}A)} = \frac{\sin A}{\cos A}.$$

13. 11. $$u = \frac{2\sin\frac{1}{2}(A + 3A)\cos\frac{1}{2}(A - 3A)}{2\cos\frac{1}{2}(A + 3A)\cos\frac{1}{2}(A - 3A)} = \frac{\sin 2A}{\cos 2A}.$$

13. 12. $$u = \frac{(\sin A + \sin 5A) + \sin 3A}{(\cos A + \cos 5A) + \cos 3A}$$
$$= \frac{2\sin\frac{1}{2}(A + 5A)\cos\frac{1}{2}(A - 5A) + \sin 3A}{2\cos\frac{1}{2}(A + 5A)\cos\frac{1}{2}(A - 5A) + \cos 3A}$$
$$= \frac{2\sin 3A\cos 2A + \sin 3A}{2\cos 3A\cos 2A + \cos 3A} = \frac{\sin 3A(1 + 2\cos 2A)}{\cos 3A(1 + 2\cos 2A)} = \tan 3A.$$

---

22. 4. By (115), area $S = rs$, and, therefore, since r is here constant, area $\propto s \propto 2s \propto a + b + c$.

22. 5. By (115 N.B.) $s - a = \alpha$, $s - b = \beta$, $s - c = \gamma$,
and $\alpha + \beta + \gamma = 3s - (a + b + c) = 3s - 2s = s$;
$$\therefore r = \frac{S}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}}.$$

22. 6. By (118) $A : A'$
$= 6r^2 \sin\frac{1}{6}\pi \times \cos\frac{1}{6}\pi : 6r^2 \sin\frac{1}{6}\pi \div \cos\frac{1}{6}\pi = \cos^2\frac{1}{6}\pi : 1 = 3 : 4.$

22. 8. Hypothenuse (c) = diameter of circumscribing circle = 30; and one side $(a) = 18$; $\therefore$ third side $(b) = \sqrt{(900-324)} = 24$: $\therefore s = 36$, $s-a = 18$, $s-b = 12$, $s-c = 6$, and area $S = \frac{1}{2}ab = 216$;
$$\therefore r = \frac{S}{s} = 6,\ r_1 = \frac{S}{s-a} = 12,\ r_2 = \frac{S}{s-b} = 18,\ r_3 = \frac{S}{s-c} = 36.$$

---

29. 3. (ii) cot 45° 0′ 35″ = tan 44° 59′ 25″; $\therefore d = \frac{25}{60} \times 2527 = 1053$; and L cot 45° 0′ 35″ = 9.9997473 + .0001053 = 9.9998526.

29. 5. L cos 30° = 10 + log-cos 30° = $10 + \log(\frac{1}{2}\sqrt{3}) = 10 + \frac{1}{2}\log 3 - \log 2$ = 9.9375306; $\therefore D = -729$:

(i) d for 20″ = $\frac{20}{60} \times -729 = -243$;
$\therefore$ L cos 30° 0′ 20″ = 9.9375306 − .0000243 = 9.9375063:

(ii) L sin 59° 59′ 25″ = L cos 30° 0′ 35″,
for which $d = \frac{35}{60} \times -729 = -425$;
$\therefore$ L sin 59° 59′ 25″ = 9.9375306 − .0000425 = 9.9374881:

(iii) L sec 30° 0′ 25″ = 20 − L cos 30° 0′ 25″:
and d for 25″ = $\frac{25}{60} \times -729 = -303$;
$\therefore$ L cos 30° 0′ 25″ = 9.9375306 − .0000303 = 9.9375003;
$\therefore$ L sec 30° 0′ 25″ = 10.0624997.

29. 6. L tan 44° 59′ = 20 − L cot 44° 59′ = 9.9997473, and L tan 45° = 10 + log-tan 45° = 10 + log (1) = 10; $\therefore D = 2527$:

(i) d for 20″ = $\frac{20}{60} \times 2527 = 842$,
and L tan 44° 59′ 20″ = 9.9997473 + .0000842 = 9.9998315:

(ii) L tan 45° 0′ 30″ = 20 − L cot 45° 0′ 30″ = 20 − L tan 44° 59′ 30″; d for 30″ = $\frac{30}{60} \times 2527 = 1264$; $\therefore$ L tan 44° 59′ 30″ = 9.9997473 + .0001264 = 9.9998737, and L tan 45° 0′ 30″ = 10.0001263:

(iii) Since L cot 44° 59′ = 10.0002527, and L cot 45° = 10,
let Log-cot^{-1}10.0001234 = 44° 59′ n″;
then $d = -1293$, $D = -2527$, and $n = \frac{1293}{2527} \times 60 = 31$;
$\therefore$ Log-cot^{-1}10.0001234 = 44° 59′ 31″.

MISCELLANEOUS EXAMPLES.

1. $A = \frac{3}{4}B = \frac{3}{2}C$;
$\therefore B = \frac{4}{3}A$, $C = \frac{2}{3}A$, and $A + B + C = A + \frac{4}{3}A + \frac{2}{3}A = 180^\circ$,

2. $A = B + 30^\circ = C - 30^g = C - \frac{9}{10} \times 30^\circ = C - 27^\circ$;
$\therefore B = A - 30^\circ,\ C = A + 27^\circ$,
and $A + B + C = A + (A - 30^\circ) + (A + 27^\circ) = 180^\circ$, &c.

3. $B = \frac{3}{4}A = C$; $\therefore A + B + C = A + \frac{3}{4}A + \frac{3}{4}A = 180^\circ$, &c.

4. Suppose a circle described round $ABCDE$: then $\angle DB$
$= \angle$ at circumference on base $AD = 2 \times (\angle$ on base $AE) = 2\times 3$
by (16) $= 72^\circ$; and $\angle BAE = \angle$ at circumference on base BCD
$= 3 \times (\angle$ on base $AE) = 3 \times 36^\circ = 108^\circ$; hence $\angle(DBA + BAE$
$= 180^\circ$, and BD is parallel to AE: similarly, CE is parallel to AB: and $\therefore BAEF$ is a ▱. Also $\angle ABF : \angle BAE$
$= 2 \times 36^\circ : 3 \times 36^\circ = 2 : 3$.

5. $a = \theta r = \frac{1}{4}\pi r = \frac{1}{4} \times 3.14159$ *mile* $= 440 \times 3.14159$ *yds* $= 1382$ *yds*.

6. $45^g = \frac{9}{10} \times 45^\circ = 40\frac{1}{2}^\circ$; $\therefore$ third angle $= 180^\circ - (45^\circ + 40\frac{1}{2}^\circ) = 94\frac{1}{2}^\circ$,
which $= \frac{10}{9} \times 94\frac{1}{2}^g = 105^g$.

7 (i) $\theta = \dfrac{a}{r} = \dfrac{\pi}{5\frac{1}{2}} = \dfrac{22}{7} \times \dfrac{2}{11} = \dfrac{4}{7}$;

(ii) $\theta = \dfrac{A}{180}\pi = \dfrac{\pi}{180}\pi = \dfrac{1}{180}\left(\dfrac{22}{7}\right)^2 = \dfrac{121}{2205}$.

8. $180^\circ - A = \frac{2}{3}(180^\circ - B) = \frac{4}{5}(180^\circ - C)$,
whence $B = \frac{3}{2}A - 90^\circ,\ C = \frac{5}{4}A - 45^\circ$,
and $A + B + C = A + (\frac{3}{2}A - 90^\circ) + (\frac{5}{4}A - 45^\circ) = 180^\circ$, &c.

9. (i) $90^\circ - \omega^\circ = 90^\circ - 57^\circ.29577 = 32^\circ.70423 = 32^\circ\ 42'\ 15''$;
(ii) $180^\circ - \omega^\circ = 122^\circ.70423 = \frac{10}{9} \times 122^g.70423 = 136^g\ 33`\ 80``$.

10. $A = \frac{5}{4}(B + C) = \frac{3}{2}(B - C)$;
$\therefore B + C = \frac{4}{5}A,\ B - C = \frac{2}{3}A$, and $B = \frac{11}{15}A,\ C = \frac{1}{15}A$;
$\therefore A + B + C = A + \frac{11}{15}A + \frac{1}{15}A = \frac{9}{5}A = 180^\circ$, or $A = 100^\circ$.

11. Let m, n, be the numbers of sides in the polygons respectively; then

(i) $\dfrac{(m-2)\,180^\circ}{(n-2)\,180^\circ} = \dfrac{m-2}{n-2} = \dfrac{7}{8}$,

(ii) $\dfrac{m-2}{m}\,180^\circ \div \dfrac{n-2}{n}\,180^\circ = \dfrac{n(m-2)}{m(n-2)} = \dfrac{35}{36}$;

(ii) $\div$ *(i)* gives $\dfrac{n}{m} = \dfrac{10}{9}$, or $n = \dfrac{10}{9}m$, which put for n in (i).

2. The two base-angles $= 180° - \pi° = 180° - 3°.14159 = 176°.85841$; $\therefore$ each of them $= 88°.42920 = 88°\ 25'\ 45''$.

3. The sum of the angles of a quadrilateral is $360°$ or 2π: and the given angles being as $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, or as 1, 2, 3, 4, (whose sum is 10,) they will be (see Colenso's *Arithmetic*, Art. 104) $\frac{1}{10} \times 2\pi$, $\frac{2}{10} \times 2\pi$, $\frac{3}{10} \times 2\pi$, $\frac{4}{10} \times 2\pi$, respectively.

4. $\theta = \dfrac{a}{r} = \dfrac{1\ in.}{1\ ft.} = \dfrac{1}{12} =$ n° of *measures* in the angle; $\therefore$ n° of *degrees* in it $= \frac{1}{12} \times 57°.29577 = 4°.77464 = 4°\ 46'\ 29''$, and n° of *grades* in it $= \frac{10}{9} \times 4.77464 = 5^g\ 30^{`}\ 52^{``}$.

5. $180° - A = 2(B - C) = \frac{1}{2}\{(90° - B) + (90° - C)\}$, whence $B - C = 90° - \frac{1}{2}A$, $B + C = 2A - 180°$; $\therefore B = \frac{3}{4}A - 45°$, $C = \frac{5}{4}A - 135°$, and $A + B + C = A + (\frac{3}{4}A - 45°) + (\frac{5}{4}A - 135°) = 180°$, &c.

6. Here $(n-2)\,180 = \{(n-1)-2\}\,200$, whence $n = 12$.

7. Here $315° = 2\pi r$, $\therefore r = 315° \div 2\pi = 315° \times \frac{7}{44} = 50°$, nearly.

8. The given angle must be the *vertical* angle, since, otherwise, the two base-angles would together $= 4$, which is $> \pi$: hence the vertical angle $= 2 \times 57°.29577 = 114°\ 35'\ 30''$, and $\therefore$ each of the base-angles $= 32°\ 42'\ 15''$.

9. Let r, s, be their radii, and their lengths $\frac{4}{3}l$, $\frac{3}{4}l$, respectively; then their values are $\frac{3}{4}l \times \pi r^2$, $\frac{4}{3}l \times \pi s^2$, which are equal; $\therefore 9r^2 = 16s^2$, or $r : s :: 4 : 3$.

0. $\tan = \frac{4}{3}$; $\therefore$ (40) $\sin = \frac{4}{5}$, $\cos = \frac{3}{5}$, and $\sin + \cos = \frac{7}{5} = 1\frac{2}{5}$; also $\text{vers} = 1 - \cos = \frac{2}{5}$, $\text{covers} = 1 - \sin = \frac{1}{5}$, and $\therefore \text{vers} = 2\ \text{covers}$.

1. The length $= 378 \times 2\pi r = 378 \times \frac{22}{7} \times 10\ in = 11880\ in = 330\ yds$.

2. Here $A + B + C = \frac{1}{4}\pi + B + \frac{1}{4} = \pi$; $\therefore B = \frac{3}{4}\pi - \frac{1}{4} = \frac{1}{4}(3\pi - 1) =$ (nearly) 2.1 *measures* $= 2.1 \times 57°.29577 = 120°.321117 = 120°\ 19'\ 16''$.

3. Answer $= 42 \times \pi r^2 = 42 \times 3.14159 \times 9\ cub.\ ft = 44\ cub.\ yds$, nearly.

4. $\cos\theta = \sqrt{(1 - \sin^2\theta)} = \sqrt{\{1 - \dfrac{4a^2b^2}{(a^2+b^2)^2}\}} = \dfrac{a^2-b^2}{a^2+b^2}$; $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{2ab}{a^2-b^2}$.

5. $\cos^2 C = \dfrac{1 - \sin^2 A\ \text{cosec}^2 B}{\cos^2 A} = \dfrac{1 - \sin^2 A\,(1 + \cot^2 B)}{\cos^2 A}$

$$= \frac{\cos^2 A - \sin^2 A\ \cot^2 B}{\cos^2 A} = 1 - \tan^2 A\ \cot^2 B;$$

$$\therefore \sin^2 C = 1 - \cos^2 C = \tan^2 A\ \cot^2 B.$$

6. $\pi° = 3°.14159$, $\pi^g = \frac{9}{10} \times \pi° = 9 \times 0°.314159 = 2°.827431$, $\pi = 180°$; $\therefore$, since the four angles together $= 360°$, we have the fourth angle $= 360° - 185°.969021 = 174°.030979 = 174°\ 1'\ 52''$ or $= \frac{10}{9} \times 174^g.030979 = 193^g\ 36^{`}\ 78^{``}$.

27. $\theta = \frac{22}{180}\pi = \frac{22}{180} \times 3.14159 = .401425\ldots.$

28. Length of arc is $2\frac{3}{4} \times 2\frac{2}{3} = \frac{22}{3}$; $\therefore 2\pi r = \frac{22}{3}$, and $r = 11 \div 3\pi = 1\frac{1}{6}$.

29. $u = (\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}})\,(\frac{2}{\sqrt{3}} + \sqrt{3}) = \frac{3}{\sqrt{3}} \times \frac{5}{\sqrt{3}} = 5.$

30. Put $\sin A = x$; then $9x^2 - \dfrac{4x^2}{1-x^2} = \dfrac{1}{2}$,

$$\text{whence } 18x^4 - 11x^2 + 1 = 0, \text{ and } x^2 = \tfrac{1}{9} \text{ or } \tfrac{1}{2};$$
$$\therefore \sin A = \pm\frac{1}{3} \text{ or } \pm\frac{1}{\sqrt{2}}.$$

31. $m(1-\cos\theta) - n\{1-\cos(\pi-\theta)\} + p\{1-\cos(\pi+\theta)\}$
$= m(1-\cos\theta) - n(1+\cos\theta) + p(1+\cos\theta) = 0$, &c.

32. Here $\pi AC^2 = \frac{1}{2}\pi AB^2 = \frac{1}{2}\pi \times 2500$;
$\therefore AC^2 = 1250$, and $AC = \sqrt{1250} = 35.4$ *yds*, nearly.

33. $A^\circ = \theta\omega^\circ = \dfrac{\text{arc}}{\text{rad}} \times \omega^\circ = \dfrac{\frac{22}{8}}{\frac{11}{8}} \times 57^\circ.29577 = 114^\circ.59154 = 114^\circ\,35'\,30''.$

34. $\sin(90^\circ + 20^\circ) = +\cos 20^\circ$; $\cos(2\times 90^\circ + 40^\circ) = -\cos 40^\circ$;
$\tan(4\times 90^\circ - 30^\circ) = -\tan 30^\circ$; $\cot(5\times 90^\circ - 10^\circ) = +\tan 10^\circ$;
$\sec(6\times 90^\circ + 10^\circ) = -\sec 10^\circ$; $\operatorname{cosec}(7\times 90^\circ + 30^\circ) = -\sec 30^\circ$.

35. Since $\operatorname{cosec} 30^\circ = 2 = \operatorname{cosec}\frac{1}{6}\pi$, $\therefore \operatorname{cosec}^{-1} 2 = 30^\circ$ or (in general terms) $= n\pi + (-1)^n\frac{1}{6}\pi = \{6n + (-1)^n\}\frac{1}{6}\pi$.

36. Here $m - \sin\theta = \cos\theta$, and $n + \cos\theta = \sin\theta$;
$$\therefore \sin\theta + \cos\theta = m, \quad \sin\theta - \cos\theta = n;$$
$$\text{hence } \sin\theta = \tfrac{1}{2}(m+n), \quad \cos\theta = \tfrac{1}{2}(m-n);$$
and $\sin^2\theta + \cos^2\theta = 1 = \frac{1}{4}\{(m+n)^2 + (m-n)^2\}$, whence $m^2 + n^2 = 2$.

37. $\tan\theta = \frac{4}{3}$;
$\therefore$ (40) $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, and $u = (\frac{4}{5} + \frac{4}{3} + \frac{5}{3})\,(\frac{3}{5} + \frac{3}{4} + \frac{5}{4}) = 9.88.$

38. $(-\cos A - \cos A) \times (-\tan A - \tan A) \times (\operatorname{cosec} A + \operatorname{cosec} A)$
$= -2\cos A \times -2\tan A \times 2\operatorname{cosec} A = 8.$

39. Here $\operatorname{vers}\theta = +\frac{1}{2}$, (the other root being impossible, since the versed-sine is never negative,) and $\therefore \cos\theta = \frac{1}{2} = \cos\frac{1}{3}\pi$; hence $\theta = 2n\pi \pm \frac{1}{3}\pi = (6n \pm 1)\,\frac{1}{3}\pi$.

40. Adding, $2\sec A = m + n$; subtracting, $2\operatorname{cosec} A = m - n$;
$$\therefore \tan A = \frac{\sec A}{\operatorname{cosec} A} = \frac{m+n}{m-n}.$$
$$\text{Also } \sec^2 A = 1 + \tan^2 A; \ \therefore \tfrac{1}{4}(m+n)^2 = 1 + \frac{(m+n)^2}{(m-n)^2},$$
$$\text{or } (m^2 - n^2)^2 = 4\{(m+n)^2 + (m-n)^2\} = 8(m^2 + n^2).$$

41. $\text{vers}(\frac{1}{2}\pi+\theta)=1-\cos(\frac{1}{2}\pi+\theta)=1+\sin\theta$; $\text{covers}(\pi+\theta)=1-\sin(\pi+\theta)$ $= 1+\sin\theta$; $\text{suvers}(\frac{3}{2}\pi+\theta) = 1+\cos(\frac{3}{2}\pi+\theta) = 1+\sin\theta$.

42. $(+\sin\theta)(-\sin\phi)=(-\text{cosec}\,\theta)(+\text{cosec}\,\phi)$, whence $\sin^2\theta=\text{cosec}^2\phi$.

43. (i) $\tan x = \pm\dfrac{1}{\sqrt{3}} = \tan\pm\frac{1}{6}\pi$; $\therefore x = n\pi\pm\frac{1}{6}\pi=(6n\pm1)\frac{1}{6}\pi$:

(ii) $\cos x = -\frac{1}{2} = \cos\frac{2}{3}\pi$; $\therefore x = (2n\pi\pm\frac{2}{3})\pi = (6n\pm2)\frac{1}{3}\pi$.

44. Let A, $B=2A$, $C=3A$, be the angles;

then $A+2A+3A=180°$, or $A=30°$, whence $C=90°$;

also $c=a+100=c\sin A+100=\frac{1}{2}c+100$;

$\therefore c=200$, $a=100$, $b=c\sin 60°=100\sqrt{3}$,

and area $=\frac{1}{2}ab=5000\sqrt{3}=8660$ *sq. yds.*

45. $(\sin A\cos C-\cos A\sin C)\cos B$

$+(\sin B\cos C-\cos B\sin C)\cos A=0$,

or (dividing every term by $\cos A\cos B\cos C$)

$\tan A-\tan C+\tan B-\tan C=0$.

46. $\cos(60°+1')+\cos(60°-1')=2\cos60°\cos1'=\cos1'$;

$\therefore \cos60°\,1'=\cos1'-\cos59°\,59'$.

47. $u=\{1+\cos(60°+A)\}-\{1-\sin(30°+A)\}$

$=\cos(60°+A)+\cos(60°-A)=2\cos60°\cos A=\cos A$.

48. $$u=\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}\times\frac{\sin A}{\cos A}=\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}\times\frac{2\sin\frac{1}{2}A\cos\frac{1}{2}A}{\cos A}=\frac{2\cos^2\frac{1}{2}A}{\cos A}$$
$$=\frac{1+\cos A}{\cos A}=\sec A+1.$$

49. $$u=\frac{2\sin\frac{1}{2}(A+2A)\cos\frac{1}{2}(A-2A)}{2\cos\frac{1}{2}(A+2A)\cos\frac{1}{2}(A-2A)}=\frac{\sin\frac{3}{2}A}{\cos\frac{3}{2}A}=\tan\tfrac{3}{2}A.$$

50. $u=\{1-\cos(A+45°)\}-\{1-\sin(B+45°)\}=\cos(45°-B)-\cos(A+45°)$

$=2\sin\frac{1}{2}\{(A+45°)+(45°-B)\}\sin\frac{1}{2}\{(A+45°)-(45°-B)\}$

$=2\sin\{45°+\frac{1}{2}(A-B)\}\sin\frac{1}{2}(A+B)$.

51. $u=(\sin^2A+\cos^2A)+(\sin^2B+\cos^2B)+2(\cos A\cos B+\sin A\sin B)$

$=1+1+2\cos(A\sim B)=2\{1+\cos(A\sim B)\}=4\cos^2\frac{1}{2}(A\sim B)$.

52. $u=1-2\cot\frac{1}{2}A+\cot^2\frac{1}{2}A=\text{cosec}^2\frac{1}{2}A-2\cot\frac{1}{2}A$

$$=\frac{1}{\sin^2\frac{1}{2}A}-\frac{2\cos\frac{1}{2}A}{\sin\frac{1}{2}A}=\frac{1-2\cos\frac{1}{2}A\sin\frac{1}{2}A}{\sin^2\frac{1}{2}A}=\frac{1-\sin A}{\frac{1}{2}(1-\cos A)}=\&\text{c.}$$

53. $$u=\frac{2\sin(2A+A)}{2\sin\frac{1}{2}(3A+A)\sin\frac{1}{2}(3A-A)}=\frac{\sin2A\cos A+\cos2A\sin A}{\sin2A\sin A}$$
$=\cot A+\cot2A$.

54. $u=(\cos^2A-\sin^2B)-(\cos^2C-\sin^2B)+(\cos^2C-\sin^2A)$

$=\cos^2A-\sin^2A=\cos2A$.

55. $$u = \frac{1+\tan\frac{1}{2}A}{1-\tan\frac{1}{2}A} + \frac{1-\tan\frac{1}{2}A}{1+\tan\frac{1}{2}A} = \frac{(1+\tan\frac{1}{2}A)^2 + (1-\tan\frac{1}{2}A)^2}{1-\tan^2\frac{1}{2}A}$$
$$= \frac{2(1+\tan^2\frac{1}{2}A)}{1-\tan^2\frac{1}{2}A} = \frac{2(\cos^2\frac{1}{2}A+\sin^2\frac{1}{2}A)}{\cos^2\frac{1}{2}A-\sin^2\frac{1}{2}A} = \frac{2}{\cos A}.$$

56. $$u = \frac{\tan A(1-\cot A)-(1+\tan A)}{(1+\tan A)(1-\cot A)} = \frac{\tan A - 1 - 1 - \tan A}{1+\tan A - \cot A - 1}$$
$$= \frac{2}{\cot A - \tan A} = \frac{2\tan A}{1-\tan^2 A} = \tan 2A.$$

57. $$u = \{1-\cos(A-B)\}\times\{1+\cos(A+B)\}$$
$$= 1 - \{\cos(A-B)-\cos(A+B)\} - \cos(A+B)\cos(A-B)$$
$$= 1-2\sin A\sin B-(\cos^2 A-\sin^2 B)=\sin^2 A-2\sin A\sin B+\sin^2 B.$$

58. $$u = \cos\{(45^\circ+A)-(45^\circ+B)\} - \cos\{(45^\circ+A)+(45^\circ+B)\}$$
$$= \cos(A-B) - \cos(90^\circ+A+B) = \cos(A-B) + \sin(A+B).$$

59. $$u = \frac{4\sin^2 A\cos^2 A - 4\sin^2 A}{4\sin^2 A\cos^2 A - 4\cos^2 A} = \frac{\sin^2 A(\cos^2 A-1)}{\cos^2 A(\sin^2 A-1)} = \frac{\sin^4 A}{\cos^4 A}.$$

60. $$u = \frac{\cos(60^\circ+A)\cos(60^\circ-A)}{\sin(60^\circ+A)\sin(60^\circ-A)} = \frac{\cos^2 60^\circ - \sin^2 A}{\cos^2 A - \cos^2 60^\circ} = \frac{\frac{1}{4}-\sin^2 A}{\cos^2 A - \frac{1}{4}}$$
$$= \frac{1-4\sin^2 A}{4\cos^2 A - 1} = \frac{1-2(1-\cos 2A)}{2(1+\cos 2A)-1} = \frac{2\cos 2A - 1}{2\cos 2A + 1} = \frac{2-\sec 2A}{2+\sec 2A}.$$

61. In the figure of (66), draw CD perpendicular on AB; then, if $BC = \frac{1}{2}AB$, we have $\sin A = \frac{1}{2}$, or $A = 30^\circ$, and $B = 60^\circ$: but $BC = CD \operatorname{cosec} 60^\circ = 60 \times \frac{2}{\sqrt{3}} = 40\sqrt{3} = 69.3$ *ft*, nearly, and $AB = 2BC = 138.6$ *ft*, and $AC = CD \operatorname{cosec} 30^\circ = 120$ *ft*; ∴ perimeter = 328 *ft*, nearly.

62. $$\cos 47^\circ + \cos 25^\circ = 2\cos\tfrac{1}{2}(47^\circ+25^\circ)\cos\tfrac{1}{2}(47^\circ-25^\circ)$$
$= 2\cos 36^\circ\cos 11^\circ$; so too, $\cos 61^\circ + \cos 11^\circ = 2\cos 36^\circ\cos 25^\circ$;
$$\therefore\ u = 2\cos 36^\circ(\cos 11^\circ - \cos 25^\circ) = 2\cos 36^\circ \times 2\sin 18^\circ\sin 7^\circ$$
$$= 4\times\tfrac{1}{4}(\sqrt{5}+1)\times\tfrac{1}{4}(\sqrt{5}-1)\times\sin 7^\circ = \sin 7^\circ.$$

63. $$u = \tan^{-1}\frac{\frac{1}{10}+\frac{9}{11}}{1-\frac{9}{110}} = \tan^{-1}\frac{101}{101} = \tan^{-1} = \tfrac{1}{4}\pi.$$

64. If θ be the circular measure of the Earth's *apparent diameter*, then $\theta = \frac{a}{r} = \frac{\text{Earth's diameter}}{\text{Sun's distance}} = \frac{1}{12{,}000}$; hence the angle in seconds $= \frac{206265}{12000} = 17''$, nearly.

65. The dip is 8 inches for a mile, and
$$d' = d\sqrt{\frac{h'}{h}} = \sqrt{\frac{486\times 12}{8}} = 27 \text{ miles}.$$

6. Here $A - B = B - C$, or $B = \frac{1}{2}(A + C)$, $C = 2B - A$;

$\therefore\ u = 2\sin\frac{1}{2}(A - C)\cos\frac{1}{2}(A + C) = 2\sin\frac{1}{2}\{A - (2B - A)\}\cos B$
$= 2\sin(A - B)\cos B.$

7. Since $\tan A = \sqrt{\dfrac{1 - \cos 2A}{1 + \cos 2A}}$, $\therefore\ \tan 22^\circ\,30' = \sqrt{\dfrac{1 - \cos 45^\circ}{1 + \cos 45^\circ}}$

$$= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}} = \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + 1}} = \sqrt{\frac{(\sqrt{2} - 1)^2}{2 - 1}} = \sqrt{2} - 1 = .4142\&c.$$

8. $\sin 30^\circ = \frac{1}{2} = .5000000$; $\therefore$ D for $60'' = 2519$;

$\therefore$ (i) d for $30'' = 1259$, and $\sin 29^\circ\,59'\,30''$
$= .4997481 + .0001259 = .4998740$;

(ii) $d = 296$; $\therefore\ n = \frac{296}{2519} \times 60 = 7$, and $A = 29^\circ\,59'\,7''$.

9. $$u = \frac{\frac{a}{b} + \frac{c}{b}}{\frac{b}{a} + \frac{c}{a}} = \frac{a}{b} \cdot \frac{a + c}{b + c} = \frac{b}{a} \cdot \frac{a^2}{b^2} \cdot \frac{a + c}{b + c} = \frac{b}{a} \cdot \frac{c^2 - b^2}{c^2 - a^2} \cdot \frac{c + a}{c + b}$$

$$= \frac{b}{a} \cdot \frac{c^2 - b^2}{c + b} \cdot \frac{c + a}{c^2 - a^2} = \&c.$$

0. The Sun's apparent diameter = (in circular measure) $\dfrac{31\frac{1}{2} \times 60}{206265}$;

$\therefore$ his actual diameter $= \theta r$

$$= \frac{31\frac{1}{2} \times 60}{206265} \times 96{,}000{,}000 = \frac{12096000000}{13751} = 879{,}645 \textit{ miles}$$

$= 880{,}000$ *miles* nearly:

hence the radius = 440,000 *miles*, which exceeds the Moon's distance by 200,000 *miles*.

1. $u = \tan^{-1}\dfrac{(2 + \sqrt{3}) - (2 - \sqrt{3})}{1 + (4 - 3)} = \tan^{-1}\sqrt{3} = \frac{1}{3}\pi = \sec^{-1} 2.$

2. By (93 Ex. 4) $\dfrac{h}{8\ in} = \left(\dfrac{177}{1}\right)^2$, whence $h = 31329 \times 8\ in = 20886\ ft.$

3. $$u = \frac{1 - \cos A}{1 - \cos B} = \frac{1 - \frac{b}{c}}{1 - \frac{a}{c}} = \frac{c - b}{c - a} = \frac{c^2 - b^2}{c + b} \times \frac{c + a}{c^2 - a^2} = \frac{a^2}{c + b} \times \frac{c + a}{b^2} = \&c.$$

4. Since the distances of the Sun from the Earth and of the Earth from the Sun are the same, their disks will appear to each other as circles, whose diameters will be proportional to their actual diameters, that is, they will be in the ratio of *880000 : 8000* = 110 : 1; hence the areas of the appare *disks will be as* $(110)^2 : 1 = 12100 : 1$.

75. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = -\dfrac{bc}{2bc} = -\dfrac{1}{2} = \cos 120°.$

76. Here $s = 440$, $s-a = 110$, $s-b = 165 = s-c$;
$\therefore$ Area $= \sqrt{\{440 \times 110 \times 165 \times 165\}} = 2 \times 110 \times 165$
$= 36300$ *sq. yds* $= 7\tfrac{1}{2}A.$

77. $$u = \frac{\sin(A+B)\sin(A-B)}{\sin(B+C)\sin(B-C)} \times \frac{\sin(B+C)}{\sin(A+B)}$$
$$= \frac{\sin^2 A - \sin^2 B}{\sin^2 B - \sin^2 C} \times \frac{\sin A}{\sin C} = \frac{(a^2-b^2)\sin A}{(b^2-c^2)\sin C} \text{ by (110).}$$

78. Here $A = 30°$, $a = 3$, $b = 3\sqrt{3}$;
$\therefore \sin B = \dfrac{b}{a} \times \sin A = \dfrac{3\sqrt{3}}{3} \times \dfrac{1}{2} = \dfrac{\sqrt{3}}{2} = \sin 60°$ or $\sin 120°$;
$\therefore$ (i) $B = 60°$, $C = 90°$, $c = a \operatorname{cosec} A = 3 \times 2 = 6$;
(ii) $B = 120°$, $C = 30° = A$; $\therefore c = a = 3$.

79. D for $60'' = -4281$; $\therefore$ (i) d for $35° = \frac{35}{60} \times -4281 = 2497$, and $\text{L}\cot 72° \, 15' \, 35'' = 9.5052819 - .0002497 = 9.5050322$:
$\text{L}\tan 17° \, 44' \, 20'' = \text{L}\cot 72° \, 15' \, 40''$;
and d for $40'' = \frac{2}{3} \times -4281 = -2854$;
$\therefore \text{L}\tan 17° \, 44' \, 20'' = 9.5052819 - .0002854 = 9.5049965.$

80. $$\text{Area} = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}(b^2+c^2)\frac{bc \sin A}{b^2+c^2}$$
$$= \tfrac{1}{2}(b^2+c^2)\frac{\sin A \sin B \sin C}{\sin^2 B + \sin^2 C} = \&\text{c, by (110).}$$

81. $B = 90° - A = 36°$; $a = c \sin A = 108 \times \frac{1}{4}(\sqrt{5}+1) = 27 \times 3.2360 = 87.4$ *ft*, nearly; $b = c \cos A = 108 \times \frac{1}{4}\sqrt{(10-2\sqrt{5})} = 27\sqrt{(10-4.4720)}$ $= 27\sqrt{5.5280} = 27 \times 2.35 = 63.5$ *ft*, nearly.

82. (i) $$u = \frac{2\sin\frac{1}{2}(A+C)\cos\frac{1}{2}(A-C)}{2\cos\frac{1}{2}(A+C)\cos\frac{1}{2}(A-C)} = \tan\tfrac{1}{2}(A+C) = \cot\tfrac{1}{2}B;$$
(ii) $$u = \frac{2\sin\frac{1}{2}(A+C)\sin\frac{1}{2}(A-C)}{2\cos\frac{1}{2}(A+C)\sin\frac{1}{2}(A-C)} = \tan\tfrac{1}{2}(A+C) = \cot\tfrac{1}{2}B.$$

83. $128 = 2^7$; $\therefore \log 128 = 7 \log 2 = 7 \times .3010300 = 2.1072100$:
$\therefore$ (i) $\log 1.28 = .1072100$; (ii) $\log .00128 = \bar{3}.1072100$;
(iii) $\log(\frac{1}{128})^3 = 3 \operatorname{colog} 128 = 3 \times \bar{3}.8927900 = \bar{7}.6783700.$

84. Area $= 25 \times 30 \times \sin 54° = \frac{375}{2}(\sqrt{5}+1) = \frac{375}{2}(3.2360) = 606\frac{3}{4}$ *sq. ft.*

85. Here $B = 60°$, $a = 50$ *ft*, $c = 100$ *ft*; $\therefore b^2 = a^2 + c^2 - 2ac \cos B$ $= 2500 + 10000 - 5000 = 7500$, or $b = 50\sqrt{3} = 50 \times 1.7320 = 86.6$ *ft*:
also $\sin C = \dfrac{c}{b}\sin B = \dfrac{100}{50\sqrt{3}} \times \dfrac{\sqrt{3}}{2} = 1$, or $C = 90°$, and $\therefore A = 30°$.

86. Area $= \frac{1}{2}bc \sin A = \frac{1}{2} \times 80 \times 125 \times \frac{1}{2}\sqrt{3} = 2500 \times 1.7320 = 4330$ *sq. yds* $= \&$c.

37. (i) $\log 125 = 3\log 5 = 2.0969100$; $\therefore \log 12.5 = 1.0969100$:
(ii) $\log 132 = \log(11\times 12) = \log(11\times 4\times 3) = \log 11 + 2\log 2 + \log 3$
$= 2.1205740$; $\therefore \log .00132 = \bar{3}.1205740$:
(iii) $\log 216 = \log(8\times 27) = 3\log 2 + 3\log 3 = 2.3344539$;
$\therefore \log 2.16 = .3344539$;
(iv) $\log 14\frac{2}{9} = \log\frac{128}{9} = \log 128 - \log 9 = 7\log 2 - 2\log 3 = 1.1529674$.

8. $\theta = \sin\theta$ (nearly) $= \frac{1}{472}$,
which angle (in *seconds*) $= \frac{206265''}{472} = 437'' = 7'\,17''$.

9. $\log\sqrt[7]{7} = \frac{1}{7}\times .8450980 = .1207282$; $\therefore d = 227$, $D = 329$, and
$n = \frac{227}{329} = .69$; $\therefore \sqrt[7]{7} = 1.320469$.

0. $h' : h :: d'^2 : d^2$; and, when $d = 1$ *mile* $= 80$ *chains*, $h = 8$ *inches*;
$\therefore h' = \frac{8}{6400}\times d'^2 = \frac{1}{800}d'^2$ *inches*, where d' must be expressed (as d has been) in *chains*.

1. Each side $(a) = \frac{20}{3}$ *chains*; $\therefore$ area $= \frac{1}{2}bc\sin A = \frac{1}{2}a^2\sin 60^\circ$
$= \frac{100}{9}\sqrt{3} = 19.244$ *sq. ch.* $= 1.9244A = 1$A 3R 28P, nearly.

2. Let A, $B = 2A$, $C = 7A$, be the angles;
then $A + 2A + 7A = 180^\circ$, or $A = 18^\circ$, $C = 126^\circ$;
$\therefore c : a = \sin C : \sin A = \sin 126^\circ : \sin 18^\circ = \sin 54^\circ : \sin 18^\circ$
$= \sqrt{5} + 1 : \sqrt{5} - 1$.

3. $\cot^{-1}2 + \cot^{-1}4 = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} = \tan^{-1}\dfrac{\frac{1}{2}+\frac{1}{4}}{1-\frac{1}{8}} = \tan^{-1}\frac{6}{7}$:

$\cot^{-1}3 + \cot^{-1}5 = \tan^{-1}\dfrac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}} = \tan^{-1}\frac{4}{7}$;

$\cot^{-1}7 + \cot^{-1}21 = \tan^{-1}\dfrac{\frac{1}{7}+\frac{1}{21}}{1-\frac{1}{147}} = \tan^{-1}\frac{14}{73}$;

and $\tan^{-1}\frac{4}{7} + \tan^{-1}\frac{14}{73} = \tan^{-1}\dfrac{\frac{4}{7}+\frac{14}{73}}{1-\frac{56}{511}} = \tan^{-1}\frac{390}{455} = \tan^{-1}\frac{6}{7}$.

4. $\theta = 1$ *foot* $\div 1$ *mile* $= \frac{1}{5280} =$ (in seconds) $\frac{206265''}{5280} = 39''$, nearly.

5. $u = \frac{1}{2}\{1 + \cos(2A + B)\} + \frac{1}{2}\{1 + \cos(2C + B)\}$
$= 1 + \frac{1}{2}\{\cos(2A + B) + \cos(2C + B)\}$
$= 1 + \frac{1}{2}\{2\cos(A + B + C)\cos(A - C)\} = 1$,
since $\cos(A + B + C) = \cos 90^\circ = 0$.

6. Let $C = 120^\circ$;
then $a = b$, and $\frac{1}{2}c = a\sin\frac{1}{2}A$, or $c = 2a\sin 60^\circ = a\sqrt{3}$;
$\therefore$ perimeter $= 2a + c = a(2 + \sqrt{3}) = 50$ *yds*;
$\therefore a = \dfrac{50}{2+\sqrt{3}} = 50(2 - \sqrt{3})$ *yds*: and area $= \frac{1}{2}ab\sin C = \frac{1}{2}a^2\sin 60^\circ$
$= 625(7 - 4\sqrt{3})\sqrt{3} = 625(7\sqrt{3} - 12) = 625\times .124 = 77\frac{1}{2}$ sq. yds.

97. Since $A+2B=(A+B)+B=(180^\circ-C)+B=180^\circ-(C-B)$, we have

$$\begin{aligned} u &= \sin(C-B)+\sin(A-C)+\sin(B-A) \\ &= 2\sin\tfrac{1}{2}(C-B+A-C)\cos\tfrac{1}{2}(C-B-A+C)-\sin(A-B) \\ &= 2\sin\tfrac{1}{2}(A-B)\{\cos\tfrac{1}{2}(2C-A-B)-\cos\tfrac{1}{2}(A-B)\} \\ &= 2\sin\tfrac{1}{2}(A-B)\times 2\sin\tfrac{1}{2}\{\tfrac{1}{2}(A-B)+\tfrac{1}{2}(2C-A-B)\} \\ &\qquad\qquad \times\sin\tfrac{1}{2}\{\tfrac{1}{2}(A-B)-\tfrac{1}{2}(2C-A-B)\} \\ &= 4\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(C-B)\sin\tfrac{1}{2}(A-C) = \text{\&c.} \end{aligned}$$

98. $AB=200$, $AC=200\cos 30^\circ=100\sqrt{3}=173.2$,
$BC=200\sin 30^\circ=100$;
$\therefore (AC+CB)-AB=73.2$ *ft*, and time of running this distance (at 4 miles $=4\times 5280$ *ft* per hour) $=\frac{73.2}{21120}$ *hour* $=\frac{549}{44}$ *seconds* $=12\tfrac{1}{2}''$, nearly.

99. $$\begin{aligned} u &= 2\cos^2\tfrac{1}{2}(A-B)+2\cos\tfrac{1}{2}(B-C+C-A)\cos\tfrac{1}{2}(B-C-C+A) \\ &= 2\cos\tfrac{1}{2}(A-B)\{\cos\tfrac{1}{2}(A-B)+\cos\tfrac{1}{2}(A+B-2C)\} \\ &= 2\cos\tfrac{1}{2}(A-B)\times 2\cos\tfrac{1}{2}\{\tfrac{1}{2}(A-B)+\tfrac{1}{2}(A+B-2C)\} \\ &\qquad\qquad \times\cos\tfrac{1}{2}\{\tfrac{1}{2}(A-B)-\tfrac{1}{2}(A+B-2C)\} \\ &= 4\cos\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A-C)\cos\tfrac{1}{2}(C-B) = \text{\&c.} \end{aligned}$$

100. In figure of (66), produce CA to D, so that $AD=100$ *ft*, $\angle BAC=60^\circ$, $\angle BDC=45^\circ$; then (41) $CD=CB=h$, suppose:
$\therefore AC=h-100$, and $BC=AC\tan 60^\circ$, or $h=(h-100)\sqrt{3}$,

$$\text{whence } h=\frac{100\sqrt{3}}{\sqrt{3}-1}=\frac{100\sqrt{3}(\sqrt{3}+1)}{3-1}=50(3+\sqrt{3})$$
$$=50\times 4.7320=237\textit{ ft}, \text{ nearly:}$$

$\therefore$ height of tower $=237+6=243$ *ft*.

Metcalfe and Palmer, Printers, Cambridge.

SOLUTIONS OF EXAMPLES

IN

PLANE TRIGONOMETRY.

PART II.

BY

THE REV. J. W. COLENSO, M.A.

RECTOR OF FORNCETT ST. MARY, NORFOLK,
LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

LONDON:
LONGMAN, BROWN, GREEN, AND LONGMANS.
1853.

BIBLIOTHECA · BODLEIANA ·

LONDON:
SPOTTISWOODES and SHAW,
New-street-Square.

SOLUTIONS,

&c.

N.B. 1. The letter u will be sometimes used, for shortness, to express the quantity given in the question.

2. The bracket ( ) refers to Articles in Part I., [ ] to Articles in Part II.

1. $\sin x = \sin a(\sin x \cos y + \cos x \sin y)$, whence, dividing by $\cos x$, $\tan x = \sin a(\tan x \cos y + \sin y)$, or $\tan x(1 - \sin a \cos y) = \sin a \sin y$.

2. By (40) we have $\sin\theta = \dfrac{b}{\sqrt{(a^2+b^2)}}$, $\cos\theta = \dfrac{a}{\sqrt{(a^2+b^2)}}$;
$\therefore \cos 2\theta = \cos^2\theta - \sin^2\theta = \dfrac{a^2-b^2}{a^2+b^2}$, $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2ab}{a^2+b^2}$:
hence $u = \dfrac{(a^3 - ab^2) + 2ab^2}{a^2+b^2} = a$.

3. By (56) $u = (2\sin\frac{3}{4}A)(2\sin\frac{1}{4}A) = 4\sin\frac{3}{4}A\sin\frac{1}{4}A =$ by (83) $4\{\sin^2\frac{1}{2}(\frac{3}{4}A + \frac{1}{4}A) - \sin^2\frac{1}{2}(\frac{3}{4}A - \frac{1}{4}A)\} = 4\sin^2\frac{1}{2}A - 4\sin^2\frac{1}{4}A$ $=$ &c. by (56).

4. $m \operatorname{vers}\theta = n \pm p(1 + \cos\theta) = n \pm p \pm p(1 - \operatorname{vers}\theta)$;
$$\therefore (m \pm p) \operatorname{vers}\theta = n \pm 2p.$$

5. Let e, e', e'', represent the *magnitudes* of an English degree, minute, and second, f, f', f'', those of a foreign grade, minute, and second: then, as in (14), $90e =$ a right angle $= 100f$, or $9e = 10f$: but $e = 60e'$, $e' = 60e''$, and $f = 100f'$, $f' = 100f''$;
hence $\dfrac{e}{f} = \dfrac{10}{9} = \dfrac{60e'}{100f'} = \dfrac{3e'}{5f'}$, or $\dfrac{e'}{f'} = \dfrac{10 \times 5}{9 \times 3} = \dfrac{2.5^2}{3.3^2}$;
so $\dfrac{2.5^2}{3.3^2} = \dfrac{e'}{f'} = \dfrac{60e''}{100f''} = \dfrac{3e''}{5f''}$, or $\dfrac{e''}{f''} = \dfrac{2.5^3}{3.3^3}$.

6. $c^2 = a^2 + b^2$, $u = \frac{1}{4}\sqrt{\{2(a^2+b^2)^2 + 2a^2b^2 - a^4 - b^4 - (a^2+b^2)^2\}}$ $= \frac{1}{4}\sqrt{(4a^2b^2)} = \frac{1}{2}ab$.

7. By (115), area $S = rs$, and $\therefore$, since r is here constant, area $\propto s \propto 2s \propto a + b + c$.

8. Treating the earth's radius as a small circular arc, we have $\theta = \frac{1}{60} =$ circular measure of the angle it subtends at the moon, which angle in *degrees* $= \frac{1}{60} \times 57.29577 = 0^\circ.95$.

9. Here $s = 15$, $s - a = 2$, $s - b = 3$, $s - c = 10$; $\therefore$ by (109) area $= \sqrt{(15 \times 2 \times 3 \times 10)} = 30$ *sq. yds.*

10. $$\frac{187}{14} \log \tfrac{1}{3} = \frac{187}{14} \times - \log 3 = -\frac{187}{14} \times .4771213 = -6.3729773$$
$$= \bar{7}.6270227 = \log(.00000042366).$$

11. chd $108^\circ =$ by (56) $2r \sin 54^\circ =$ by (100) $2r \times \frac{1}{4}(\sqrt{5}+1)$ $= 2r\{\frac{1}{4}(\sqrt{5}-1)+\frac{1}{2}\} = 2r \sin 18^\circ + 2r \sin 30^\circ =$ chd $36^\circ +$ chd 60°.

12. $$\frac{\cos B}{\cos A} = \frac{1}{\cos C};\ \therefore \frac{\cos B - \cos A}{\cos B + \cos A} = \frac{1 - \cos C}{1 + \cos C},$$
$$\text{or } \frac{2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)} = \frac{2 \sin^2 \frac{1}{2} C}{2 \cos^2 \frac{1}{2} C}.$$

13. By (80), $u = \frac{1}{2}\{\cos(m+n)\theta + \cos(m-n)\theta\} \cos r\theta$
$$= \tfrac{1}{2} \cos(m+n)\theta \cos r\theta + \tfrac{1}{2} \cos(m-n)\theta \cos r\theta$$
$$= \tfrac{1}{4}\{\cos(m+n+r)\theta + \cos(m+n-r)\theta\}$$
$$+ \tfrac{1}{4}\{\cos(m-n+r)\theta + \cos(m-n-r)\theta\}.$$
Observe that in the *Answer* $\cos(n+r-m)\theta$ is the same as $\cos(m-n-r)\theta$ in the *above*, since $\cos A = \cos(-A)$.

14. In fig. (1) and (2) of Art. 105, $\tan A = \frac{CD}{AD} = \frac{CD}{AB \mp BD}$ $= \frac{a \sin B}{c - a \cos B}$ in both cases, and in fig. (3), $\tan(180^\circ - A)$ $= -\tan A = \frac{CD}{AD} = \frac{CD}{BD - AB} = \frac{a \sin B}{a \cos B - c}$; hence, in each case, $\tan A = \frac{a \sin B}{c - a \cos B}$; and, since, in like manner, $\tan A$ $= \frac{a \sin C}{b - a \cos C}$,
$$\therefore \cot A = \frac{b - a \cos C}{a \sin C} = \frac{b}{a} \operatorname{cosec} C - \cot C.$$

15. $$\cot^2 \tfrac{1}{2} A = \frac{2 \cos^2 \frac{1}{2} A}{2 \sin^2 \frac{1}{2} A} = \frac{1 + \cos A}{1 - \cos A} = \frac{1 + \dfrac{1}{\sec A}}{1 - \dfrac{1}{\sec A}} = \frac{\sec A + 1}{\sec A - 1}.$$

16. By (83), $u = \frac{1}{2}(\cos 72^\circ + \cos 2A) + \frac{1}{2}(\cos 108^\circ + \cos 2A)$ $= \cos 2A$, since $\cos 108^\circ = -\cos 72^\circ$.

17. $\frac{\sin^2}{1-\sin^2}+4\sin^2-1=0$, or $4\sin^4-6\sin^2+1=0$; $\therefore \sin^2$
$=\frac{6\pm2\sqrt{5}}{8}$, &c.

3. $BC=AC\tan A$; $\log BC=\log 200+L\tan A-10=(2+\log 2)$
$+(20-L\cot A)-10=2+(1-\log 5)+10-L\tan 64°50'$
$=13-.6989700-10.3280372=1.9729928=\log 93.97$.

19. By (78), $\sin A+\cos A=-\sqrt{(1+\sin 2A)}$, $\sin A-\cos A=$
$+\sqrt{(1-\sin 2A)}$, &c.

By (83), $u=1+\cos^2\tfrac{1}{2}\{(2A-2B)+2B\}-\sin^2\tfrac{1}{2}\{2A-2B)-2B\}$
$=1+\cos^2A-\sin^2(A-2B)=$&c.

21. Here $\sin A\cos B\cos C+\cos A\sin B\cos C=2\cos B\cos C\sin A$
$+2\sin B\sin C\sin A$; $\therefore$ (dividing by $\sin A\sin B\cos C$)
$\cot B+\cot A=2\cot B+2\tan C$, or $\cot A-\cot B=2\tan C$.

in $(A-B)=\sin\{A-(90°-A)\}=\sin(2A-90°)=-\sin(90°-2A$
$=-\cos 2A$; $\sin 2A+\sin 2B=$ by (81) $2\sin(A+B)\cos(A-B)$
$=2\cos(A-B)$, since $\sin(A+B)=\sin 90°=1$.

23. By (115 N.B.), $\alpha=s-a$, $\beta=s-b$, $\gamma=s-c$, and $\alpha+\beta+\gamma$
$=3s-(a+b+c)=s$;

$$\therefore r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}=\sqrt{\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}}.$$

24. By (27), $\theta=\frac{42}{180}\pi=\frac{7}{30}\times 3.14159=.73303\ldots$

25. By (83), $u=\tfrac{1}{2}\{\sin(A+B-C)+\sin(A-B+C)\}-$
$\tfrac{1}{2}\{\sin(B+A-C)+\sin(B-A+C)\}$
$=\tfrac{1}{2}\{\sin(A-B+C)-\sin(B-A+C)\}$
$=\tfrac{1}{2}\{\sin C+(A-B)\}-\sin\{C-(A-B)\}$
$=\sin(A-B)\cos C$, by (80).

26. $\sin 285°=\sin(270°+15°)=-\cos 15°=$ by (71 Ex. 1.),
$-\frac{\sqrt{3}+1}{2\sqrt{2}}=-\tfrac{1}{4}(\sqrt{6}+\sqrt{2})$; $\sec 285°=+\operatorname{cosec} 15°=\frac{2\sqrt{2}}{\sqrt{3}-1}$
$=\frac{2\sqrt{2}(\sqrt{3}+1)}{3-1}=\sqrt{6}+\sqrt{2}$; $\tan 285°=-\cot 15°$
$=-\frac{\sqrt{3}+1}{\sqrt{3}-1}=-\frac{(\sqrt{3}+1)^2}{3-1}=-(2+\sqrt{3})$.

27. Here $\tan^{-1}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\tan^{-1}\frac{(\sqrt{3}+\sqrt{2})^2}{3-2}=\tan^{-1}(5+2\sqrt{6})$;
$\tan^{-1}\sqrt{\tfrac{3}{2}}=\tan^{-1}\tfrac{1}{2}\sqrt{6}$; and by (86), $\tan^{-1}(5+2\sqrt{6})+$

$$\tan^{-1}\tfrac{1}{2}\sqrt{6}=\tan^{-1}\frac{(5+2\sqrt{6})+\frac{1}{2}\sqrt{6}}{1-(5+2\sqrt{6})\frac{1}{2}\sqrt{6}}=\tan^{-1}\frac{10+5\sqrt{6}}{-10-5\sqrt{6}}$$
$$=\tan^{-1}(-1)=\tfrac{3}{4}\pi.$$

28. $u=\left(1+\dfrac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}\right)^2=\dfrac{(\cos\frac{1}{2}A+\sin\frac{1}{2}A)^2}{\cos^2\frac{1}{2}A}=$ by (73) and (76) $\dfrac{1+\sin A}{\frac{1}{2}(1+\cos A)}=$ &c.

29. $u=7(4\log 2-\log 3-\log 5)+5(2\log 5-\log 3-3\log 2)$
$+3(4\log 3-3\log 2-1)=4\log 2+3\log 5-3$
$=4\log 2+3(1-\log 2)-3=\log 2.$

30. Here $A-B=B-C$, or $B=\frac{1}{2}(A+C)$; and $\dfrac{\sin A-\sin C}{\cos C-\cos A}$
$=$ by (81) $\dfrac{2\sin\frac{1}{2}(A-C)\cos\frac{1}{2}(A+C)}{2\sin\frac{1}{2}(A-C)\sin\frac{1}{2}(A+C)}=\dfrac{\cos B}{\sin B}.$

31. Here $\dfrac{\tan\theta+1}{1-\tan\theta}+\dfrac{\tan\theta-1}{1+\tan\theta}=\dfrac{4\tan\theta}{1-\tan^2\theta}=2\tan 2\theta=$ (by question) $2\tan 60^\circ$, $\therefore 2\theta=60^\circ$, or (64. iii.) $=n\pi+\frac{1}{6}\pi.$

32. Height $=70\tan 60^\circ=70\sqrt{3}$; $\therefore \log(\text{height})=\log 70+\frac{1}{2}\log 3$
$=1.8450980+.2385606=2.0836586=\log 121.$

33. Subtract 1 from each side; then $\dfrac{\cos A-\cos(B-C)}{\cos B\cos C}$
$=\dfrac{\cos D-\cos(E-F)}{\cos E\cos F}$; $\therefore \dfrac{\cos B\cos C}{\cos E\cos F}=\dfrac{\cos A-\cos(B-C)}{\cos D-\cos(E-F)}$
$=\dfrac{(1-\text{vers}\,A)-(1-\text{vers}\,(B-C))}{(1-\text{vers}\,D)-(1-\text{vers}\,(E-F))}=$ &c.

34. $u=2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$
$=2\sin\frac{1}{2}C\{\cos\frac{1}{2}A\cos\frac{1}{2}B+\sin\frac{1}{2}A\sin\frac{1}{2}B\}$
$=2\sin\frac{1}{2}C\left\{\sqrt{\dfrac{s(s-a)}{bc}}\sqrt{\dfrac{s(s-b)}{ac}}+\sqrt{\dfrac{(s-b)(s-c)}{bc}}\times\sqrt{\dfrac{(s-a)(s-c)}{ac}}\right\}$ by (107)
$=2\sin\frac{1}{2}C\left\{\dfrac{s}{c}\sqrt{\dfrac{(s-a)(s-b)}{ab}}+\dfrac{s-c}{c}\sqrt{\dfrac{(s-a)(s-b)}{ab}}\right\}$
$=2\sin\frac{1}{2}C\left\{\dfrac{2s-c}{c}\sin\frac{1}{2}C\right\}=\dfrac{2(a+b)}{c}\sin^2\frac{1}{2}C.$

35. Here $25000=2\pi r$, or $2r=\dfrac{25000}{\pi}=\dfrac{2500}{18}\times\dfrac{180}{\pi}$
$=\dfrac{1250}{9}\times 57.29577=7958$ miles.

36. $u=\dfrac{1}{\cos(\frac{1}{4}\pi+\theta)\cos(\frac{1}{4}\pi-\theta)}=(80)\dfrac{1}{\cos^2\frac{1}{4}\pi-\sin^2\theta}=\dfrac{1}{\frac{1}{2}-\sin^2\theta}$
$=\dfrac{2}{1-2\sin^2\theta}=\dfrac{2}{\cos 2\theta}$.

37. Let $A=15°$; then, since $B+C=180°-A$, we have $2\frac{2}{3}C+C$ $=180°-15°$, or $\frac{11}{3}C=165°$; $\therefore C=\frac{3}{11}\times 165°=45°$, and $B=120°$; hence
$b : c :: \sin B : \sin C :: \frac{1}{2}\sqrt{3} : \frac{1}{2}\sqrt{2} :: \sqrt{3} : \sqrt{2}$.

38. Here $s=\frac{27}{2}$, $s-a=\frac{9}{2}=s-b=s-c$; $\therefore$ by (109) area $=\sqrt{(\frac{27}{2}.\frac{9}{2}.\frac{9}{2}.\frac{9}{2})}=\frac{81}{4}\sqrt{3}=35$ *sq. yds.*

39. $AB^2=108^2+144^2=36^2(9+16)$; $\therefore AB=36\times 5=180$ *yds.*; and $CD.AB=2$ area $ABC=AC.BC$, whence CD
$=108\times 144\div 180=86.4$ *yds.*; also $\dfrac{BD}{BC}=\cos B=\dfrac{BC}{AB}$,
or $BD=\dfrac{BC^2}{AB}=\dfrac{144\times 144}{180}=115.2$ *yds*; and $AD=180-115.2$ $=64.8$ *yds.*

0. Given $A+B=75°$, $A-B=75^g=\frac{9}{10}\times 75°=67\frac{1}{2}°$;
$\therefore 2A=142\frac{1}{2}°,\ 2B=7\frac{1}{2}°$,
and $A=71\frac{1}{4}°=71°\,15'$, or $=\frac{10}{9}\times 71\frac{1}{4}{}^g=79\frac{1}{6}{}^g=79^g\,16`\,66``$,
$B=3\frac{3}{4}°=3°\,45'$, or $=\frac{10}{9}\times 3\frac{3}{4}{}^g=4\frac{1}{6}{}^g=4^g\,16`\,66``$.

1. Since π would be the circular measure of 200°,
$\therefore 36°=\frac{36}{200}\pi=\frac{9}{50}\pi$ by (27).

2. Here $\sin A=\sqrt{(1-.36)}=\sqrt{.64}=.8$, $\cot A=\dfrac{.6}{.8}=\dfrac{3}{4}$,
chd $A=2\sin\frac{1}{2}A=\sqrt{(2-2\cos A)}=\sqrt{(2-1.2)}=\sqrt{.8}$ $=.894$.

3. $u=\dfrac{1}{\cos 2A}-\dfrac{\sin 2A}{\cos 2A}=\dfrac{1-\sin 2A}{\cos 2A}=\dfrac{(\cos A-\sin A)^2}{\cos^2 A-\sin^2 A}$ by (73) $=$ &c.,
striking out the factor $\cos A-\sin A$ from numr. and denr.

44. $u=(\sin\alpha+\sin\beta)-\{\sin(\alpha+\beta+\gamma)-\sin\gamma\}$
$=2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)-2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta+2\gamma)$
$=2\sin\frac{1}{2}(\alpha+\beta)\{\cos\frac{1}{2}(\alpha-\beta)-\cos\frac{1}{2}(\alpha+\beta+2\gamma)\}$
$=2\sin\frac{1}{2}(\alpha+\beta)\{2\sin(\alpha+\gamma)\sin(\beta+\gamma)\}$ by (81).

5. Here $s=27$, $s-a=13\frac{1}{2}$, $s-b=9$, $s-c=4\frac{1}{2}$;
$\therefore$ area $=\sqrt{(27\times\frac{27}{2}\times 9\times\frac{9}{2})}=\frac{27}{2}\times 9=121\frac{1}{2}$ *sq. yds.*

6. Since $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$;
$\therefore \sin 30°\,1'+\sin 29°\,59'=2\sin 30°\cos 1'=\cos 1'$.

7. $A=\frac{5}{6}\times 57°.29577=47°\,44'\,47''$

48. $u=\dfrac{(\cos\frac{1}{2}A+\sin\frac{1}{2}A)^2}{(\cos\frac{1}{2}A-\sin\frac{1}{2}A)^2}$ by (73) $=\left(\dfrac{1+\tan\frac{1}{2}A}{1-\tan\frac{1}{2}A}\right)^2$

$=$ by (71. Ex. 2) $\tan^2\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)=$ &c.

49. Here (53) sine $=\dfrac{\text{Sine}}{\text{rad}}=\dfrac{4\frac{1}{2}}{9}=\frac{1}{2}$, and so the angle is 30° or $\frac{1}{6}\pi$;

$\therefore$ area of sector $=\frac{1}{2}\theta r^2$ by (31) $=\frac{81}{12}\pi=21$ *sq. ft.* nearly.

50. Here $s=600$, $s-a=220$, $s-b=180$, $s-c=200$; $\therefore$ area $=\sqrt{(600\times220\times180\times200)}=12000\sqrt{33}=68935$ *sq. yds.* (nearly) $=78$ A. 1 R. 13 P.

51. $\tan\frac{1}{2}A=$ by (76) $\sqrt{\dfrac{1-\cos A}{1+\cos A}}=\sqrt{\dfrac{2\sin A-2\sin A\cos A}{2\sin A+2\sin A\cos A}}=$ &c.

52. Here $\theta=\dfrac{a}{r}=\dfrac{1\,in.}{3\,miles}=\dfrac{1}{190080}=$ (in *seconds*) $\dfrac{206265}{190080}$ by (26) $=1''$ nearly.

53. Let $\operatorname{cosec}\theta=\sqrt{5}$; then $\cot\theta=\sqrt{(\operatorname{cosec}^2\theta-1)}=2$; and $\therefore u=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\frac{1}{4}\pi$, as in (86 *Ex.* 2.).

54. $AE=c\cos A$, $BF=a\cos B$, $CD=b\cos C$;
$AF=b\cos A$, $BD=c\cos B$, $CE=a\cos C$:
$\therefore AE.BF.CD=abc\cos A\cos B\cos C=AF.BD.CE.$

55. $u=2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)+2\cos\frac{1}{2}(\alpha+\beta+2\gamma)\cos\frac{1}{2}(\alpha+\beta)$
$=2\cos\frac{1}{2}(\alpha+\beta)\{\cos\frac{1}{2}(\alpha-\beta)+\cos\frac{1}{2}(\alpha+\beta+2\gamma)\}$
$=2\cos\frac{1}{2}(\alpha+\beta)\{2\cos\frac{1}{2}(\alpha+\gamma)\cos\frac{1}{2}(\beta+\gamma)\}.$

56. A, B, the points of observation; C, the tree; then $\angle CAB=\alpha=\angle CBA$, and (if CD be drawn perp. to AB) $CD=AD\tan\alpha=\frac{1}{2}a\tan\alpha$.

57. By 127 (2), area $=\frac{1}{2}(750+1225)\times1540=15{,}20750$ *sq. links* $=15$ A. 0 R. $33\frac{1}{5}$ P.

58. $u=\dfrac{1-\tan\frac{1}{2}\theta}{1+\tan\frac{1}{2}\theta}+\dfrac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}=\dfrac{2(1+\tan^2\frac{1}{2}\theta)}{1-\tan^2\frac{1}{2}\theta}=\dfrac{2}{\cos\theta}$ by (76).

59. $\dfrac{\sin\alpha}{\tan\alpha}=\dfrac{m\sin\beta}{n\tan\beta}$, or $\cos\alpha=\dfrac{m}{n}\cos\beta$; $\therefore\sin^2\beta+\cos^2\beta=1$

$=\dfrac{\sin^2\alpha}{m^2}+\dfrac{n^2\cos^2\alpha}{m^2}$, and $m^2=(1-\cos^2\alpha)+n^2\cos^2\alpha$, &c.

60. Since $\cos C=-\cos(A+B)$, &c.

$\therefore u=\dfrac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}=$ by (80)

$\dfrac{1-(\cos^2A-\sin^2B)}{1-(\cos^2A-\sin^2C)}=\dfrac{\sin^2A+\sin^2B}{\sin^2A+\sin^2C}=\dfrac{a^2+b^2}{a^2+c^2}$ by (110).

61. Since π would be the circular measure of 120°, $\therefore 75^\circ = \frac{75}{120}\pi = \frac{5}{8}\pi$.

62. Since $\sec^2 = 1 + \tan^2$, $\therefore \frac{s^2 s'^2 - s^2}{s^2 s'^2 - s'^2} = \frac{s^2(s'^2-1)}{s'^2(s^2-1)} = \frac{(1+t^2)t'^2}{(1+t'^2)t^2}$ $=$ &c.

63. By (40), $\sin\theta = \frac{2\sqrt{(ab)}}{\sqrt{\{4ab+(a-b)^2\}}} = \frac{2\sqrt{(ab)}}{a+b}$;

$\therefore \cos\theta = \sqrt{(1-\sin^2\theta)} = \frac{a-b}{a+b}$; $\therefore$ vers $\theta = 1 - \frac{a-b}{a+b} = \frac{2b}{a+b}$.

64. log 72 = log 8 + log 9 = 3 log 2 + 2 log 3 = &c. : log 45 = log 9 + log 5 = 2 log 3 + (1 − log 2) = &c. : log 1800 = log 2 + log 9 + log 100 = log 2 + 2 log 3 + 2 : log .00024 = log (3 × 8) − log 100000 = log 3 + 3 log 2 − 5 = 1.3802113 − 5 = $\bar{4}$.3802113.

65. $u = \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \frac{2\cos 2\theta}{\sin 2\theta} = 2\cot 2\theta =$ &c. by (76).

66. Sin 48° = sin (30° + 18°) = sin 30° cos 18° + cos 30° sin 18° = &c. by (100).

67. area $= \frac{1}{2}bc\sin A = \frac{1}{2}(a^2+b^2+c^2)\frac{bc\sin A}{a^2+b^2+c^2} =$ &c. (110. Ex. 3.)

68. $CA = CP$ cosec α, $CB = CQ$ cosec β;

$\therefore a = r$ (cosec β − cosec α), or $r =$ &c

69. $u = \frac{1}{a(\cos^2\frac{1}{2}x + \sin^2\frac{1}{2}x) + b(\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x)}$

$= \frac{1}{(a+b)\cos^2\frac{1}{2}x + (a-b)\sin^2\frac{1}{2}x} =$ &c., dividing numr. and denr. by $\cos^2 x$.

70. $R + r = \frac{1}{2}a\left(\text{cosec}\frac{\pi}{n} + \cot\frac{\pi}{n}\right)$ by (117) $= \frac{1}{2}a\frac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}$

$= \frac{1}{2}a\cot\frac{\pi}{2n}$ by (74)

71. $\frac{\sin ACD}{\sin ADC} = \frac{\sin BCD}{\sin BDC}$, or $\frac{AD}{AC} = \frac{BD}{BC}$, (as appears also from Euc. 6. 3;) hence $a^2 \times AD^2 = b^2 \times BD^2$, or,

$a^2(b^2 + CD^2 - 2b\,.\,CD\cos\frac{1}{2}C) = b^2(a^2 + CD^2 - 2a\,.\,CD\cos\frac{1}{2}C)$;

$\therefore (a^2-b^2)CD^2 = 2(a^2b - ab^2)CD\cos\frac{1}{2}C = 2ab(a-b)CD\cos\frac{1}{2}C$,

and $CD =$ &c.

72. The *log in question* is −1.8753145 = $\bar{2}$.1246855; hence d, the *diff. of this* from the lowest of the two given logs, = 183, and D,

the diff. of the two given logs $= 326$; hence (143 Ex. 2) $n = \frac{182}{326} = .56$, and $\therefore$ the sequence of figures for the required number is 1332556, and, since the characteristic is $\bar{2}$, we have the number itself $= .01332556$.

73. $2\cos\theta = \surd(2 + 2\cos 2\theta) = \surd\{2 + \surd(2 + 2\cos 4\theta)\}$;
$\therefore \sec\theta =$ &c.

74. $\frac{p^2}{qr} + \frac{q^2}{pr} + \frac{r^2}{pq} = \frac{b\sin C \cdot c\sin B}{a\sin C \cdot a\sin B} + \frac{a\sin C \cdot c\sin A}{b\sin C \cdot b\sin A} + \frac{a\sin B \cdot b\sin A}{c\sin B \cdot c\sin A}$
$=$ &c.

75. $\{1 - \cos(2\pi - \theta)\} - \{1 - \sin(\frac{3}{2}\pi - \theta)\} - \{1 + \cos(\pi - \theta)\} + \{1 + \cos(\theta - 2\pi)\} + \{1 - \sin(\theta - \frac{3}{2}\pi)\} - \{1 + \cos(\theta - \pi)\}$
$= (1 - \cos\theta) - (1 + \cos\theta) - (1 - \cos\theta) + (1 + \cos\theta) + (1 - \cos\theta) - (1 - \cos\theta) = 0$.

76. area $= \frac{1}{2} bc\sin A = \frac{1}{2}(a + b + c)^2 \frac{bc\sin A}{(a + b + c)^2}$

$= \frac{1}{2}(a + b + c)^2 \frac{\sin A \sin B \sin C}{(\sin A + \sin B + \sin C)^2}$ by (110)

$= \frac{1}{2}(a + b + c)^2 \frac{8\sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C \cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C}{16\cos^2\frac{1}{2}A \cos^2\frac{1}{2}B \cos^2\frac{1}{2}C}$

(as in Ex. 16.4) $=$ &c.

77. $A = 57°.29577 \times .7854 = 45°$ nearly.

78. Let $n + 1$, n be the numbers of sides; then (15) $\frac{(n + 1) - 2}{n + 1} 180$

$= \frac{n - 2}{n} 180 + 4$, or $\frac{n - 1}{n + 1} - \frac{n - 2}{n} = \frac{4}{180}$, that is, $\frac{2}{n(n + 1)} = \frac{1}{45}$, or

$n^2 + n = 90$, whence $n = 9$

79. $a\sin\theta + b\cos\theta = \frac{ab^{\frac{1}{3}}}{\surd(a^{\frac{2}{3}} + b^{\frac{2}{3}})} + \frac{ha^{\frac{1}{3}}}{\surd(a^{\frac{2}{3}} + b^{\frac{2}{3}})} = \frac{a^{\frac{1}{3}} b^{\frac{1}{3}} (a^{\frac{2}{3}} + b^{\frac{2}{3}})}{\surd(a^{\frac{2}{3}} + b^{\frac{2}{3}})}$

$=$ &c.: $a\sec\theta + b\,\mathrm{cosec}\,\theta = \frac{a\surd(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{a^{\frac{1}{3}}} + \frac{b\surd(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{b^{\frac{1}{3}}}$

$= (a^{\frac{2}{3}} + b^{\frac{2}{3}})\surd(a^{\frac{2}{3}} + b^{\frac{2}{3}}) =$ &c.

80. By (27) $\theta = \frac{A}{200}\pi = \frac{1.0375}{200}\pi = .0051875 \times 3.14159 = .016297$ nearly; which angle in *degrees* $= \frac{9}{10} \times 1.0375 = .93375$ $= 56' 1\frac{1}{2}''$.

81. Here $\frac{1}{3}(2\pi r) = 2\pi r'$, or $r = 3r'$; and, if n be the number of *degrees* required of the smaller circle, then the length of

$n^\circ = \frac{n}{180} \times \pi r' = r$, by question; $\therefore n = \frac{r}{r'} \times \frac{180}{\pi} = 3 \times 57.29,577$ $= 171.88731$, and the arc required is 171° 53′ 14″.

2. Here $s = 21$, $s - a = 8$, $s - b = 7$, $s - c = 6$; $\therefore$ area $=$ $\sqrt{(21 \times 8 \times 7 \times 6)} = 84$ *sq. chains* $= 8,40000$ *sq. links* $= 8$ A. 1 R. 24 P.

3 $$u = \left\{1 - \cos\left(\frac{m}{m+n}\pi - \theta\right)\right\} + \left\{1 - \cos\left(\frac{n}{m+n}\pi + \theta\right)\right\}$$
$$= 2 - \cos\left(\frac{m}{m+n}\pi - \theta\right) + \cos\left\{\pi - \left(\frac{n}{m+n}\pi + \theta\right)\right\}$$
$$= 2 - \cos\left(\frac{m}{m+n}\pi - \theta\right) + \cos\left(\frac{m}{m+n}\pi - \theta\right) = 2.$$

$$u = (\sin 41^\circ + \sin 67^\circ) - (\sin 31^\circ + \sin 77^\circ)$$
$$= 2 \sin 54^\circ \cos 13^\circ - 2 \sin 54^\circ \cos 23^\circ$$
$$= 2 \sin 54^\circ (\cos 13^\circ - \cos 23^\circ) = 4 \sin 54^\circ \sin 18^\circ \sin 5^\circ$$
$$= 4 \times \frac{\sqrt{5} + 1}{4} \times \frac{\sqrt{5} - 1}{4} \sin 5^\circ = \sin 5^\circ.$$

Let d be the diameter of the given circle, d', d'', of the two smaller circles; then $d' : d'' :: 1\frac{2}{3} : 2\frac{1}{2} :: 2 : 3$, and $\therefore d' : d' + d''$ (or d) $:: 2 : 5$, and $d'' : d :: 3 : 5$; hence the areas are as $d^2 : d'^2 : d''^2 = 5^2 : 2^2 : 3^2$.

$A + B = m$ *degrees* $= \frac{10}{9} m$ *grades*,
$A - B = n$ *grades* $= \frac{9}{10} n$ *degrees*;
$\therefore 2A = (m + \frac{9}{10} n)$ *degrees*, $2B = (\frac{10}{9} m - n)$ *grades*, &c.

$$\frac{\sin ABO}{\sin BAO} \cdot \frac{\sin BCO}{\sin CBO} \cdot \frac{\sin CAO}{\sin ACO} = \frac{AO}{BO} \cdot \frac{BO}{CO} \cdot \frac{CO}{AO} = 1.$$

AB, the height of steeple, $= x$, $BC = 100$ *ft*, $BD = 225$ *ft*; $\angle ADB = \theta$, $\angle ACB = 2\theta$:

then $\tan 2\theta = \frac{x}{100}$, $\tan\theta = \frac{x}{225}$; but $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$;

$\therefore \frac{x}{100} = \frac{\frac{2}{225}x}{1 - \frac{1}{50625}x^2}$, or $1 - \frac{1}{50625}x^2 = \frac{8}{9}$,
whence $x^2 = \frac{50625}{9} = 5625$, and $x = 75$ *ft*.
Or thus: since $\angle ACB$ is double of $\angle ADB$, we have $\angle CAD$ $\angle ADC$, or $CA = CD = 125$; $\therefore \cos ACB = \frac{100}{125} = \frac{4}{5}$, and $AB = AC \sin ACB = 125 \times \frac{3}{5} = 75$ *ft*.

$2\tan A = m + n$, $2\sin A = m - n$; $\therefore \cos A = \frac{\sin A}{\tan A} = \frac{m - n}{m + n}$:

and $\sin^2 A + \cos^2 A = 1 = \frac{(m-n)^2}{4} + \frac{(m-n)^2}{(m+n)^2}$,
or $(m^2 - n^2)^2 = 4\{(m+n)^2 - (m-n)^2\} = 16mn$.

90. Here $\sin(x+a)-\sin(x-a)=\cos(x-a)-\cos(x+a)$, or $2\sin a\cos x=2\sin a\sin x$;
$\therefore\ \tan x=1=\tan 45^\circ$, or $x=n\pi+\frac{1}{4}\pi$, by (64.iii).

91. $u=1-\dfrac{\sin^3 A}{\sin A+\cos A}-\dfrac{\cos^3 A}{\cos A+\sin A}=1-\dfrac{\sin^3 A+\cos^3 A}{\sin A+\cos A}$
$=1-(\sin^2 A-\sin A\cos A+\cos^2 A)=\sin A\cos A=\frac{1}{2}\sin 2A.$

92. Let $a=$ side of square, $r=$ rad. of each circle $=\frac{1}{4}a$; then area required $=a^2-4\pi r^2=a^2-\frac{1}{4}\pi a^2=a^2(1-.78539$, &c.
$=.21460\,sq.\,ft.=31\ sq.\ in.$

93. Here $s=120$, $s-a=50$, $s-b=40$, $s-c=30$; $\therefore$ area $=\sqrt{(120\times 50\times 40\times 30)}=1200\sqrt{5}=2683\ sq.\ yds$
$=2$ R. 8 P. 21 *sq. yds.*

94. The largest angle will be opposite to the greatest side: let this be $c=5$; then $\tan\frac{1}{2}C=\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}=\sqrt{\dfrac{3\times 2}{6\times 1}}=1$;
$\therefore\ \frac{1}{2}C=45^\circ$, and $C=90^\circ$.

95. The two values of B (112) being B and $180^\circ-B$, we have $A+B+C=180^\circ$ and $A+(180^\circ-B)+C'=180^\circ$; hence, by adding, we get $2A+C+C'=180^\circ$,
or $A=90^\circ-\frac{1}{2}(C+C')$, &c.

96. Here $A=6^\circ.21$, and $\theta=\dfrac{6.21}{180}\pi=\dfrac{69}{2000}\pi$; $\therefore a=\theta r=\dfrac{69}{20}\pi\ yds$
$=\dfrac{69}{20}\times\dfrac{22}{7}=10.84\,yds=32\frac{1}{2}\,ft.$

97. The circumference traced by its extremity $=2\pi r=14\pi\,yds$ hence in 1′ it describes $140\pi=140\times\frac{22}{7}\ yds=440\ yds$
$=\frac{1}{4}\,mile$, and in 1 *h.* it describes 15 *miles.*

98. $u=\dfrac{\sin(A+30^\circ)\sin(A-30^\circ)}{\cos(A+30^\circ)\cos(A-30^\circ)}=\dfrac{\sin^2 A-\sin^2 30^\circ}{\cos^2 A-\sin^2 30^\circ}$
$=\dfrac{\frac{1}{2}(1-\cos 2A)-\frac{1}{4}}{\frac{1}{2}(1+\cos 2A)-\frac{1}{4}}=$ &c.

99. $AE=AB+\frac{1}{2}BC=AB+\frac{1}{2}(AC-AB)=\frac{1}{2}(AB+AC)$
$=\frac{1}{2}(72^\circ+120^\circ)=96^\circ$; $BCDF=360^\circ-BAF=360^\circ-2\times 72^\circ=216^\circ$; $\angle BAC=\frac{1}{2}$ angle at centre on $BC=\frac{1}{2}BC$
$=\frac{1}{2}(AC-AB)=24^\circ$; $\angle BCA=\frac{1}{2}$ angle at centre on $AB=\frac{1}{2}AB=36^\circ$; and $\angle ABC=180^\circ-24^\circ-36^\circ=120^\circ$.

100. The iron required will be that contained in the diff. of two *cylinders, the* length of each being the width of the roller, *that is, 3½ ft,* and the radius of the inner cylinder being ½ inch

less than that of the outer, r, suppose. Now the circumf. of the outer being $5\frac{1}{2} ft$, we have $2\pi r = 5\frac{1}{2} ft = 66 in$, or $r = \frac{33}{\pi}$ $= \frac{33}{\frac{22}{7}} = 10\frac{1}{2} in$; hence the lesser radius $(r') = 9\frac{1}{2} in$. Consequently area of section of cylinder $\pi r^2 - \pi r'^2 = \frac{22}{7} . (\frac{441}{4} - \frac{361}{4})$ $= \frac{440}{7} sq. in$, and multiplying this by the length $(= 3\frac{1}{2} ft = 42 in,)$ we have $\frac{440}{7} \times 42 = 2640$ *cub. in.* = quantity of iron required.

101. $2\cos A = \surd(2 + 2\cos 2A) = \surd(2 + \surd(2 + 2\cos 4A)) =$ &c.
$2\sin A = \surd(2 - 2\cos 2A) =$ &c.

102. By (75) $x = 4\cos^3\phi, y = 4\sin^3\phi$;
$\therefore 1 = \cos^2\phi + \sin^2\phi = (\frac{1}{4}x)^{\frac{2}{3}} + (\frac{1}{4}y)^{\frac{2}{3}}$, &c.

103. $4z^2 - 4z^4 = \sin^2 A$, or $4z^4 - 4z^2 + 1 = \cos^2 A$;
$\therefore 2z^2 = 1 \pm \cos A = 2\cos^2 \frac{1}{2}A$, or $2\sin^2 \frac{1}{2}A$, &c.

04. $\tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta} = \frac{\tan\phi\,(a + b\cos\phi) - b\sin\phi}{(a + b\cos\phi) + \tan\phi\, b\sin\phi}$,

$\left(\text{by writing for } \tan\theta, \text{ its value } \frac{b\sin\phi}{a + b\cos\phi}\right)$

$$= \frac{a\tan\phi}{a + b(\cos\phi + \sin\phi\tan\phi)} = \frac{a\sin\phi}{a\cos\phi + b}.$$

05. $u = 2a^2\sin B\cos B + 2b^2\sin A\cos A$

$$= \frac{2a^2b}{c}\sin C\cos B + \frac{2b^2a}{c}\sin C\cos A$$

$$= \frac{ab}{c^2}\sin C\,(2ac\cos B + 2bc\cos A)$$

$$= \frac{ab\sin C}{c^2}\{(a^2 + c^2 - b^2) + (b^2 + c^2 - a^2)\} = \&c.$$

06. $u = \frac{1}{2}\{\sin(\alpha + (\beta - \gamma)) + \sin(\alpha - (\beta - \gamma))\} -$
$\frac{1}{2}\{\sin(\beta + (\alpha - \gamma)) + \sin(\beta - (\alpha - \gamma))\}$
$= \frac{1}{2}\{\sin(\gamma + (\alpha - \beta)) - \sin(\gamma - (\alpha - \beta))\} = \sin(\alpha - \beta)\cos\gamma$.

07. Let l, m, n, be the perpendiculars from A, B, C, on sides a, b, c, respectively; then (115) $rs = \text{area } ABC = \frac{1}{2}al = \frac{1}{2}bm$ $= \frac{1}{2}cn$; $\therefore r^3s^3 = \frac{1}{8}abclmn$, and $lmn = \frac{(2s)^3}{abc}r^3$.

08. (1) $d = 27$, D for $10'' = 124$: $\therefore n = \frac{27}{124} \times 10 = 2.176$,

and $A' = 59° 37' 42''.18$.

(2) $n=5.34$; $\therefore\ d=\dfrac{5.34}{10}\times 124=66$,

and $L\sin A''=9.9358987$.

109. Let $\sin\theta=\sqrt{\dfrac{x}{a+x}}$; then $\cos^2\theta=1-\dfrac{x}{a+x}=\dfrac{a}{a+x}$;

and $\tan^2\theta=\dfrac{\sin^2\theta}{\cos^2\theta}=\dfrac{x}{a}$, $\cos 2\theta=\cos^2\theta-\sin^2\theta=\dfrac{a-x}{a+x}$;

$\therefore\ \theta=\tan^{-1}\sqrt{\dfrac{x}{a}}$, and $2\theta=\cos^{-1}\dfrac{a-x}{a+x}$.

110. $\sin\{m\pi+(\frac{1}{2}\pi+\theta)\}=(-1)^m\sin(\frac{1}{2}\pi+\theta)=(-1)^m\cos\theta$; $\cos\{m\pi+(\frac{1}{2}\pi+\theta)\}=(-1)^m\cos(\frac{1}{2}\pi+\theta)=(-1)^m(-\sin\theta)$ $=(-1)^{m+1}\sin\theta$; $\tan\{m\pi+(\frac{1}{2}\pi+\theta)\}=\tan(\frac{1}{2}\pi+\theta)$ $=-\cot\theta$.

111. By (80), $u=\sin^2\theta-\sin^2\frac{1}{4}\pi=\sin^2\theta-\frac{1}{2}=\sin^2\frac{1}{6}\pi=\frac{1}{4}$; $\therefore\ \sin\theta=\frac{1}{2}\sqrt{3}=\sin 60^\circ$; $\therefore\ \theta=60^\circ$, or (64. i) $=n\pi+(-1)^n\frac{1}{3}\pi$.

112. From $\log\frac{5}{4}=\log\frac{10}{8}=\log 10-\log 8=1-3\log 2$, we get $\log 2$ $=.3010300$; and from $\log .\dot{1}=\log\frac{1}{9}=-2\log 3$, we get $\log 3$ $=.4771213$; and then $\log 2\frac{1}{2}=\log\frac{5}{2}=\log\frac{10}{4}=1-2\log 2$, $\log 2\frac{1}{4}=\log\frac{9}{4}=2\log 3-2\log 2$, $\log .\dot{2}=\log\frac{2}{9}=\log 2-2\log 3$.

113. $\angle BAC=\frac{1}{2}\angle BOC$, (if O be the centre of the circle,)

$=\frac{1}{2}(\angle AOC-\angle AOB)=\frac{1}{2}\left(\dfrac{360^\circ}{7}-\dfrac{360^\circ}{8}\right)=3\dfrac{3}{14}^\circ$;

and, $\angle BOC$ being $6\dfrac{3}{7}^\circ$, the number of sides in the polygon of which BC is a side will (16) be $360\div 6\dfrac{3}{7}=56=n$, suppose; $\therefore\ \angle$ of polygon $=\dfrac{n-2}{n}\times 180^\circ=173\dfrac{4}{7}^\circ$.

114. $u=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\dfrac{\sin(\alpha+\beta)\sin(\alpha+\beta)}{\cos(\alpha+\beta)\cos(\alpha+\beta)}$

$=\dfrac{\sin^2\alpha-\sin^2\beta}{\frac{1}{2}(\sin 2\alpha-\sin 2\beta)}$ by (80).

115. Sector $COB=\frac{1}{2}\theta r^2$ by (31) $=\frac{1}{2}(\alpha-\sin\alpha)r^2$, (if α be the circular measure of $\angle AOB$, or the measure of arc AB to radius unity,) $=\frac{1}{2}\alpha r^2-\frac{1}{2}r^2\sin\alpha$
$=$ sector $AOB-$ triangle AOB (109) $=$ segment ACB.

116. $u=\dfrac{2a\sin A+2b\sin B+2c\sin C}{(b^2+c^2-a^2)+(a^2+c^2-b^2)+(a^2+b^2-c^2)}$

$$=\frac{2\left(a\frac{a}{c}\sin C+b\frac{b}{c}\sin C+c\sin C\right)}{a^2+b^2+c^2}$$

$=\frac{2\sin C}{c}=\frac{2p}{ab}$, since $\frac{1}{2}cp=\text{area}\,ABC=\frac{1}{2}ab\sin C$.

By (114, 115) $Rr=\frac{abc}{4S}\times\frac{S}{s}=\frac{abc}{2(a+b+c)}$

$=\frac{bc}{2}\,\frac{\sin A}{\sin A+\sin B+\sin C}$, by (110);

∴ area $=\frac{1}{2}bc\sin A=Rr(\sin A+\sin B+\sin C)$.

$\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}=\frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)}$, whence $\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{\cos(\alpha-\beta)+\cos(\alpha+\beta)}$

$=\frac{\sin(\gamma+\delta)-\sin(\gamma-\delta)}{\sin(\gamma+\delta)+\sin(\gamma-\delta)}$, or $\frac{2\sin\alpha\sin\beta}{2\cos\alpha\cos\beta}$

$=\frac{2\cos\gamma\sin\delta}{2\sin\gamma\cos\delta}$; ∴ $\tan\delta=\tan\alpha\tan\beta\tan\gamma$.

$\sin A=\sin(B+C)=\sin B\cos C+\cos B\sin C$, (1)
$\sin B=\sin(A+C)=\sin A\cos C+\cos A\sin C$; (2)
mult. (1) by $\cos A$, and (2) by $\cos B$, and subtracting, we get
$\sin A\cos A-\sin B\cos B=\sin B\cos A\cos C-\sin A\cos B\cos C$,
or $\sin A(\cos B\cos C+\cos A)=\sin B(\cos A\cos C+\cos B)$, &c.

$$\frac{\sin A}{\sin B}=\frac{\cos B+\cos A\cos C}{\cos A+\cos B\cos C}$$

$$=\frac{\frac{a^2+c^2-b^2}{2ac}+\frac{b^2+c^2-a^2}{2bc}\times\frac{a^2+b^2-c^2}{2ab}}{\frac{b^2+c^2-a^2}{2bc}+\frac{a^2+c^2-b^2}{2ac}\times\frac{a^2+b^2-c^2}{2ab}}$$

$$=\frac{a\{2b^2(a^2+c^2-b^2)+b^4-(a^2-c^2)^2\}}{b\{2a^2(b^2+c^2-a^2)+a^4-(b^2-c^2)^2\}}=\frac{a}{b}.$$

$c^2=a^2+b^2-2ab\cos A=(a^2+b^2)-2ab(1-2\sin^2\frac{1}{2}A)$
$=(a-b)^2+4ab\sin^2\frac{1}{2}A=(a-b)^2\{1+\tan^2\phi\}$,

$\left(\text{if }\tan\phi=\frac{2\sqrt{(ab)}}{a-b}\sin\frac{1}{2}A,\right)=(a-b)^2\sec^2\phi$:

∴ $c=(a-b)\sec\phi$.

When $a=b$ nearly, $a-b$ will be very small, and ∴ $\tan\phi$ will be very great; and ϕ cannot be found very *accurately from* $\tan\phi$: see (104).

122. $u=\frac{\tan A+1}{1-\tan A}+\frac{\tan A-1}{1+\tan A}=\frac{4\tan A}{1-\tan^2 A}=2\tan 2A=2\tan 60^\circ$:

$\therefore\ 2A=60^\circ$, or $n\pi+\frac{1}{3}\pi$; and $A=30^\circ$, or $(3n+1)\frac{1}{6}\pi$.

123. $u=(1=\cos^2\theta)^{-2}\times(1+\tan^2\theta)^{-2}=\sin^{-4}\theta\times\sec^{-4}\theta=\cot^4\theta$.

124. $u=2\cos(n-1)A\cos A=\cos A$:

$\therefore$ (i) $\cos A=0$, and (ii) $\cos(n-1)A=\frac{1}{2}$:

hence (i) $A=90^\circ$, or $2n\pi\pm\frac{1}{2}\pi$;

(ii) $(n-1)A=60^\circ$, or $2n\pi\pm\frac{1}{3}\pi$.

125. $u=\{\sin(\beta+\theta)+\sin(2\alpha-\beta+\theta)\}+\sin\alpha\sin(\beta+\alpha)\sin(\beta-\alpha)$

$=2\sin(\alpha+\theta)\cos(\alpha-\beta)-\sin\alpha\sin(\alpha+\beta)\sin(\alpha-\beta)$

$=$ (by question) $\sin(\alpha+\beta)\sin(\alpha-\beta)\sin(\alpha+2\theta)$;

$\therefore\ 2\sin(\alpha+\theta)\cos(\alpha-\beta)$

$=\sin(\alpha+\beta)\sin(\alpha-\beta)\{\sin(\alpha+2\theta)+\sin\alpha\}$

$=\sin(\alpha+\beta)\sin(\alpha-\beta)\{2\sin(\alpha+\theta)\cos\theta\}$:

$\therefore\ \sec\theta=\sin(\alpha+\beta)\tan(\alpha-\beta)$.

126. $\log\sqrt{\tfrac{1}{2}}=\frac{1}{2}\operatorname{colog}2=\frac{1}{2}(\bar{1}.6989700)=\bar{1}.8494850$; $\log\sqrt[3]{\tfrac{1}{3}}$

$=\frac{1}{3}\operatorname{colog}3=\frac{1}{3}(\bar{1}.5228787)=\bar{1}.8409595$; $\log\left(\tfrac{16}{5}\right)^{-\frac{1}{3}}$

$=\log\left(\tfrac{5}{16}\right)^{\frac{1}{3}}=\frac{1}{3}(\log 5-\log 16)=\frac{1}{3}\{(1-\log 2)-4\log 2\}$

$=\frac{1}{3}(1-5\log 2)=\frac{1}{3}(1-1.50515)=\frac{1}{3}(-.50515)$

$=\frac{1}{3}(\bar{1}.49485)=\bar{1}.8316166$.

127. Let A, B, C denote the three angles whose tangents are here concerned: then $A+B+C=\frac{2^n-1}{2^n}\pi+\frac{\alpha+\beta+\gamma}{2^n}=\pi$:

$\therefore$ by (85) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

128. Here, by question, $A+4A+A=180^\circ$; $\therefore\ A=30^\circ$, and since $\angle BED=\angle BDE$, $\therefore AD=EC$ or $AC=2a+b$;

hence $AB=AC\frac{\sin ACB}{\sin ABC}=(2a+b)\frac{\sin 30^\circ}{\sin 120^\circ}=\frac{2a+b}{\sqrt{3}}$.

And $b=DE=2BF\cot BDE$, (if $BF\perp AC$,)

$=2AF\tan 30^\circ\cot BDE=2(a+\frac{1}{2}b)\tan 30^\circ\cot BDE$;

$\therefore\ \tan BDE=\frac{2a+b}{b\sqrt{3}}=\frac{AB}{DE}$.

129. In the figure of Art. (105) $AB=AD+BD$ in fig. (1), $=AD-BD$ in fig. (2), $=BD-AD$, in fig. (3), which in each case will give us $c=a\cos B+b\cos A$

$=a\cos B\pm\sqrt{(b^2-b^2\sin^2 A)}=a\cos B\pm\sqrt{(b^2-a^2\sin^2 B)}$.

30. $\log 5 = \log 10 - \log 2 = \&c.$; $\log \tfrac{16}{1000} = \log 16 - \log 1000$
$= 4 \log 2 - 3 = \&c.$; $\log \tfrac{1}{2} = -\log 2 = -.30103 = \bar{1}.69897$;
$\log \tfrac{625}{100} = 4 \log 5 - 2 = 4(1 - \log 2) - 2 = \&c.$;
$\log \tfrac{8}{5} = 3 \log 2 - \log 5 = 3 \log 2 - (1 - \log 2) = \&c.$;
$\log \tfrac{125}{8} = 3 \log 5 - 3 \log 2 = 3(1 - \log 2) - 3 \log 2 = \&c.$

31. Let $\alpha = \cot^{-1}(x-1)$, $\beta = \cot^{-1}(x+1)$; $\therefore \alpha - \beta = \tfrac{1}{12}\pi = 15°$;
and $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta} = \dfrac{\cot\beta - \cot\alpha}{\cot\alpha \cot\beta + 1}$

$$= \frac{(x+1) - (x-1)}{(x^2 - 1) + 1} = \frac{2}{x^2} = \tan 15° = \tan(60° - 45°) = \frac{\sqrt{3} - 1}{\sqrt{3} + 1};$$

$$\therefore\ x^2 = \frac{2(\sqrt{3} + 1)}{\sqrt{3} - 1} = \frac{2(\sqrt{3} + 1)^2}{3 - 1} = (\sqrt{3} + 1)^2.$$

32. $u = \left(\dfrac{\sin A}{\cos A} + \dfrac{1}{\cos A}\right) \div \left(\dfrac{\cos A}{\sin A} + \dfrac{1}{\sin A}\right) = \dfrac{1 + \sin A}{\cos A} \times \dfrac{\sin A}{1 + \cos A}$
$= \tan(45° + \tfrac{1}{2}A) \tan \tfrac{1}{2}A$, by (73 and 74).

33. $\dfrac{n}{m} = \dfrac{2 \sin(a - x) \cos(a - x)}{2 \sin x \cos x} = \dfrac{\sin(2a - 2x)}{\sin 2x}$;

$$\therefore\ \frac{\sin(2a - 2x) - \sin 2x}{\sin(2a - 2x) + \sin 2x} = \frac{n - m}{n + m} = \frac{2 \cos a \sin(a - 2x)}{2 \sin a \cos(a - 2x)}$$

$= \dfrac{\tan(a - 2x)}{\tan a}$; or $\tan(a - 2x) = \dfrac{n - m}{n + m} \tan a$, &c.

34. By (118), $3 : 4 :: nr^2 \sin\dfrac{\pi}{n} \cos\dfrac{\pi}{n} : nr^2 \sin\dfrac{\pi}{n} \div \cos\dfrac{\pi}{n} :: \cos^2\dfrac{\pi}{n} : 1$;

$\therefore\ \cos\dfrac{\pi}{n} = \dfrac{\sqrt{3}}{2} = \cos\dfrac{\pi}{6}$, and $n = 6$.

35. $\dfrac{a + b}{a - b} = \dfrac{\tan(\theta + \phi)}{\tan(\theta - \phi)} = \dfrac{\sin(\theta + \phi) \cos(\theta - \phi)}{\cos(\theta + \phi) \sin(\theta - \phi)} = \dfrac{\sin 2\theta + \sin 2\phi}{\sin 2\theta - \sin 2\phi}$;

$\therefore\ \dfrac{a}{b} = \dfrac{\sin 2\theta}{\sin 2\phi}$: hence $b \sin 2\theta = a \sin 2\phi$, and $b \cos 2\theta = c - a \cos 2\phi$,
by question;
$\therefore\ b^2(\sin^2 2\theta + \cos^2 2\theta) = b^2 = a^2 \sin^2 2\phi + (c - a \cos 2\phi)^2 = \&c.$

36. The greatest angle being 108°, we have greatest side
$= \dfrac{\sin 108°}{\sin 18°} \times 96 = 96 \cot 18° = 96 \times 3.077 = 295$ ft. nearly; for $\cot 18°$

$$= \frac{\sqrt{(10 + 2\sqrt{5})}}{\sqrt{5} - 1} = \sqrt{\frac{10 + 2\sqrt{5}}{(\sqrt{5} - 1)^2}} = \sqrt{\frac{5 + \sqrt{5}}{3 - \sqrt{5}}} = \sqrt{\frac{20 + 8\sqrt{5}}{9 - 5}}$$

$= \sqrt{(5 + 2\sqrt{5})} = \sqrt{9.472136} = 3.077.$

137. $\tan\theta = \frac{1-\sin a}{\cos a} = \tan(\frac{1}{4}\pi - \frac{1}{2}a)$ by (73);
$\therefore \theta = n\pi + (\frac{1}{4}\pi - \frac{1}{2}a)$.

138. $\sin A - (1-\cos A) = (\sin A + \cos A) - 1 = \sqrt{(1+\sin 2A)} - 1$, the root being positive, since $A < 90°$, and $\therefore$ $\sin A$ and $\cos A$ are both positive; and greatest value of this is when $\sin 2A = 1$, (or $2A = 90°$,) when $A = \sqrt{2} - 1$.

139. (i) for *sign*, by (78), $\sin\theta + \cos\theta$ is $(+)$ in the first quadrant, and two adjacent half-quadrants, &c.
(ii) for *magnitude*, $\sin\theta + \cos\theta = \sqrt{(1+\sin 2\theta)}$, and as 2θ changes from 0 to 90°, or θ from 0 to 45°, this changes from 1 to $\sqrt{2}$; and then, as 2θ changes from 90° to 180°, or θ from 45° to 90°, it changes from $\sqrt{2}$ to 1; as 2θ changes from 180° to 270°, or θ from 90° to 135°, it changes from 1 to 0, and as 2θ changes from 270° to 360°, or θ from 135° to 180°, it changes from 0 to -1; and so from -1 to $-\sqrt{2}$, from $-\sqrt{2}$ to -1, from -1 to 0, from 0 to 1, as θ changes from 180° to 225°, from 225° to 270°, from 270° to 315°, from 315° to 360°.

140. Area $= \frac{1}{2}(a+b)c = \frac{1}{2}ac(1+\tan^2\theta)$, (if $b = a\tan^2\theta$,) $= \frac{1}{2}ac\sec^2\theta$.

141. $\tan\frac{1}{2}\phi = \frac{1}{\sin\phi} - \sin\phi = \frac{\cos^2\phi}{\sin\phi}$; or $\cos^2\phi = \sin\phi\tan\frac{1}{2}\phi$
$= 2\sin^2\frac{1}{2}\phi = 1 - \cos\phi$; $\therefore \cos^2\phi + \cos\phi - 1 = 0$, whence we get $\cos\phi = \frac{1}{2}(-1 \pm \sqrt{5})$, of which only the root with $(+)$ is possible, since $\cos\phi$ must, in *magnitude*, be < 1.

142. Here $A, 2B, 2C, D$, are in *A.P.*; $\therefore A + 2C = 4B, 2B + 4D = 4C$; but $A + C = 180°$, and $B + D = 180°$, from which four equations we get A, B, C, D.

143. $1 \pm \sin\theta = 1 \pm \cos(\frac{1}{2}\pi - \theta) =$ (with upper sign)
$2\cos^2\frac{1}{2}(\frac{1}{2}\pi - \theta) = 2\cos^2(\frac{1}{4}\pi - \frac{1}{2}\theta) = 2\sin^2(\frac{1}{4}\pi + \frac{1}{2}\theta)$,
or $=$ (with lower sign) $2\sin^2\frac{1}{2}(\frac{1}{2}\pi - \theta)$.

144. $1 + 2\sin 4A = 4\sin 3A\cos A$
$= 2\{\sin(3A + A) + \sin(3A - A)\} = 2\sin 4A + 2\sin 2A$;
$\therefore \sin 2A = \frac{1}{2}$, and $2A = 30°$, or $n\pi + (-1)^n\frac{1}{6}\pi$.

145. By (81), $\sin^2\frac{1}{2}(2a + \theta + \theta) - \sin^2\frac{1}{2}(2a + \theta - \theta) + n\cos^2 a = 0$
$= \sin^2(a+\theta) - \sin^2 a + n\cos^2 a$;
$\therefore (n+1)\cos^2 a = 1 - \sin^2(a+\theta) = \cos^2(a+\theta)$,
and $\sec a = \sqrt{(n+1)}\sec(a+\theta)$ and $\therefore > \sqrt{(n+1)}$, *since (48)* $\sec(a+\theta)$ must be > 1.

146. chd $\theta = \cos\theta$, or $2\sin\frac{1}{2}\theta = 1 - 2\sin^2\frac{1}{2}\theta$, whence we get $\sin\frac{1}{2}\theta = \frac{1}{2}(\sqrt{3}-1)$, or $\cos^2\frac{1}{2}\theta = 1 - \frac{1}{4}(4 - 2\sqrt{3}) = \frac{1}{2}\sqrt{3} = \cos\frac{1}{6}\pi$: $\therefore \frac{1}{2}\theta = \cos^{-1}\sqrt{(\cos\frac{1}{6}\pi)}$.

147. AB the tower, C, D the objects; $\angle ACB = \alpha$, $\angle ADB = \beta$; then $a = CD = BD - BC = AB\cot\beta - AB\cot\alpha$,

$$\text{or } AB = \frac{a}{\cot\beta - \cot\alpha} = \frac{a\sin\alpha\sin\beta}{\sin(\alpha-\beta)}.$$

$$\text{Or thus: } AC = CD\frac{\sin ADC}{\sin CAD} = a\frac{\sin\beta}{\sin(\alpha-\beta)},$$

and $AB = AC\sin\alpha = $ &c.

148. Here $\tan\frac{1}{2}\alpha\sin\alpha = 1 = 2\sin^2\frac{1}{2}\alpha$;
$\therefore \sin\frac{1}{2}\alpha = \pm\frac{1}{2}\sqrt{2} = \pm\sin 45°$, and $\frac{1}{2}\alpha = n\pi \pm \frac{1}{4}\pi$.

149. $$u = \left(\frac{\tan A + 1}{1 - \tan A} + \frac{\tan A - 1}{1 + \tan A}\right) + \left(\frac{1 - \tan A}{1 + \tan A} + \frac{1 + \tan A}{1 - \tan A}\right)$$

$$= \frac{4\tan A}{1 - \tan^2 A} + \frac{2(1 + \tan^2 A)}{1 - \tan^2 A} = \frac{2\tan A}{1 + \tan^2 A} = 2\tan A\sec^2 A$$
$$= \sin^2 A.$$

150. $m = \dfrac{\cos^2\theta}{\sin\theta}$, $n = \dfrac{\sin^2\theta}{\cos\theta}$; $\therefore \tan^3\theta = \dfrac{n}{m}$, and $mn = \sin\theta\cos\theta$

$$= \frac{n^{\frac{1}{3}}}{\sqrt{(m^{\frac{2}{3}} + n^{\frac{2}{3}})}} \times \frac{m^{\frac{1}{3}}}{\sqrt{(m^{\frac{2}{3}} + n^{\frac{2}{3}})}};\ \therefore m^{\frac{2}{3}}n^{\frac{2}{3}}(m^{\frac{2}{3}} + n^{\frac{2}{3}}) = 1.$$

151. $x - a + c(\sin l - \cos l\tan\delta) = 0, y - b + c(\cos\delta + \sin l\tan\delta) = 0$:
$\therefore (x-a)\sin l + (y-b)\cos l + c = 0$.

152. $$\beta = (\text{in seconds})\ 206265 \times 4a^2 = 206265 \times 4 \times \left(\frac{227}{206265}\right)^2$$

$$= \frac{4 \times 51529}{206265} = \frac{206116}{206265} = 1'', \text{nearly.}$$

153. (1) $2A = 2n\pi \pm A$; $\therefore A = 2n\pi$ or $\frac{2}{3}n\pi$.

(2) $\dfrac{2\tan A}{1 - \tan^2 A} = 3\tan A$; whence (i) $\tan A = 0$, or (ii) $\tan A$ $= \pm\dfrac{1}{\sqrt{3}}$; hence (i) $A = n\pi$, or (ii) $A = n\pi \pm \frac{1}{6}\pi$.

154. Let the circles touch in p, q, r; then $s = Aq + Br + Cp$, and $Aq = s - Br - Cp = s - Bp - Cp = s - a$: &c.

155. By (93 Ex. 4) $d^2 = 2rh$, or $h = \dfrac{2500}{7964}$ miles $= \dfrac{2500 \times 440}{1991}$ yards $=$ *1657 feet*, nearly.

156. $u = 1 + \cos^2 A - 2\cos A + \sin^2 A = 2\,(1 - \cos A) = 4\sin^2 \tfrac{1}{2}A$
$= \&c.$

157. Here $2\sin^2 \tfrac{1}{2}a = 2\sin^2 (x - \tfrac{1}{2}a)$;
$\therefore x - \tfrac{1}{2}a = n\pi \pm \tfrac{1}{2}a$, and $x = n\pi + a$ or $n\pi$.

158. In fig. of (19), $\sin AOD = \dfrac{\tfrac{1}{2}AB}{AO} = \dfrac{1}{2m}$;

$\therefore \cos AOB = 1 - 2\sin^2 \tfrac{1}{2}AOB = 1 - \dfrac{1}{2m^2}$.

159. $\sin 18° = \tfrac{1}{4}(\sqrt{5} - 1) = .3090169$:
$\therefore$ tab.sin $18° = 3090.169$, and D for $1' = 2.767$;
$\therefore d$ for $23'' = \tfrac{23}{60} \times 2.767 = 1.0607$, &c.

160. $u = \cos^2\theta + \sin\theta\,(\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta)^2 \dfrac{\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta}{\cos\tfrac{1}{2}\theta}$
$+ \dfrac{\sin\tfrac{1}{2}\theta + \cos\tfrac{1}{2}\theta}{\sin\tfrac{1}{2}\theta}$
$= \cos^2\theta + 2\sin^2\tfrac{1}{2}\theta\,(\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta)\,(\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta)$
$= \cos^2\theta + 2\sin^2\tfrac{1}{2}\theta\cos\theta = \cos^2\theta + (1 - \cos\theta)\cos\theta = \cos\theta$.

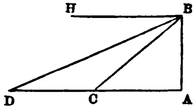
161. The radius of the circle is the *chord* of 120° to a radius of 6 *inches*, and $\therefore =$ (in inches) $6 \times \text{chd}\,120° = 12\sin 60° = 6\sqrt{3}$:
$\therefore$ area $= \pi r^2 = \tfrac{22}{7} \times 108 = 339\tfrac{3}{7}$ *sq. in.*

162. $\tan A \tan 2A + 1 = 2$, or $\tan A \tan 2A = 1 = \dfrac{2\tan^2 A}{1 - \tan^2 A}$;

$\therefore \tan^2 A = \tfrac{1}{3}$, and $A = \pm 30°$ or $n\pi \pm \tfrac{1}{6}\pi$.

163. This is to prove that $(2r\sin\tfrac{1}{5}\pi)^2$
$= (2r\sin\tfrac{1}{6}\pi)^2 + (2r\sin\tfrac{1}{10}\pi)^2$, or $\sin^2 36°$
$= \sin^2 30° + \sin^2 18°$, or by (100) $\tfrac{1}{16}(10 - 2\sqrt{5})$
$= \tfrac{1}{4} + \tfrac{1}{16}(6 - 2\sqrt{5})$, which is manifestly true.

164.



$AB = 210$, $\angle HBC = \angle BCA = 25° = \theta$,
$\angle HBD = \angle BDA = 23° = \phi$:
then $AC = AB\cot\theta$, $AD = AB\cot\phi$;
$\therefore CD = AB\,(\cot\phi - \cot\theta)$
$= 210\dfrac{\sin(\theta - \phi)}{\sin\theta\sin\phi}$, and

$\log CD = \log 210 + L\sin 2° - L\sin 25° - L\sin 23° + 10$
$= (\log 3 + \log 7 + 1) + L\sin 2° + \&c. = 1.6472122 = \log 44.3$.

165. $\tan\phi + 2\dfrac{1 - \tan^2\phi}{2\tan\phi} = \sin\phi\left(1 + \dfrac{2\tan^2\tfrac{1}{2}\phi}{1 - \tan^2\tfrac{1}{2}\phi}\right)$, or

$\dfrac{1}{\tan\phi} = \sin\phi\dfrac{1 + \tan^2\tfrac{1}{2}\phi}{1 - \tan^2\tfrac{1}{2}\phi} = \sin\phi \times \dfrac{1}{\cos\phi}$ by (76) $= \tan\phi$;
$\therefore \tan^2\phi = 1$, and $\phi = \pm 45°$ or $n\pi \pm \tfrac{1}{4}\pi$.

66. $u=(\cos^2 a-\sin^2 a)(\cos^4 a+\cos^2 a\sin^2 a+\sin^4 a)$
$=\cos 2a\{(\cos^2 a+\sin^2 a)^2-\sin^2 a\cos^2 a\}=$ &c.

67. $u=\frac{a}{2\sin A}\sin 2A+\frac{b}{2\sin B}\sin 2B+\frac{c}{2\sin C}\sin 2C$
$=R(\sin 2A+\sin 2B+\sin 2C)$ by (114)
$=R\{2\sin(A+B)\cos(A-B)+2\sin C\cos C\}$
$=2R\sin C\{\cos(A-B)-\cos(A+B)\}$,
$(\because \cos C=-\cos(A+B),)=2R\sin C\cdot 2\sin A\sin B.$

68. $\text{Cos}\,\frac{1}{2}B=\sqrt{\frac{s(s-b)}{ac}}=\sqrt{\frac{\frac{15}{2}\cdot\frac{5}{2}}{4\cdot 6}}=\frac{5}{4\sqrt{2}}$;
$\therefore L\cos\frac{1}{2}B=10+\log 5-\log 4-\frac{1}{2}\log 2=10+(1-\log 2)$
$-2\log 2-\frac{1}{2}\log 2=11-\frac{7}{2}\log 2=9.9463950$:

and here $D=-669$, $d=-90$, and $n=\frac{d}{D}\times 60=8$,

and $\therefore \frac{1}{2}B=27^\circ 53' 8''$, or $B=55^\circ 46' 16''$.

69. $\text{Cos}\,67\frac{1}{2}^\circ=\frac{1}{2}\sqrt{(2+2\cos 135^\circ)}$ by (76) $=\frac{1}{2}\sqrt{(2-\sqrt{2})}$;
$\therefore \sec 67\frac{1}{2}^\circ\frac{2}{\sqrt{(2-\sqrt{2})}}=\frac{2\sqrt{(2+\sqrt{2})}}{\sqrt{(4-2)}}=\sqrt{2}\sqrt{(2+\sqrt{2})}$
$=\sqrt{(4+2\sqrt{2})}$.

70. $u=\frac{\sin 2a+\cos 2a}{(\cos a-\cos 3a)+(\sin 3a-\sin a)}$

$$=\frac{\sin 2a+\cos 2a}{2\sin a\sin 2a+2\sin a\cos 2a}=\frac{1}{2\sin a}.$$

71. $u=(a+b\cos a)\cos\theta-b\sin a\sin\theta=k\cos(\theta+\beta)$
if $k\cos\beta=a+b\cos a$, $k\sin\beta=b\sin a$;
$\therefore k^2=k^2(\cos^2\beta+\sin^2\beta)=(a+b\cos a)^2+b^2\sin^2 a$

$=a^2+2ab\cos a+b^2$, and $\tan\beta=\frac{k\sin\beta}{k\cos\beta}=\frac{b\sin a}{a+b\cos a}$.

72. A, B, the observers, C the balloon, CD the vertical;
$AB=1$ *mile*, $\angle CAD=30^\circ$, $\angle CBD=75^\circ$;
$\therefore \angle ACB=75^\circ=\angle ABC$, and $AC=AB=1$ *mile*;
hence $CD=AC\sin 30^\circ=\frac{1}{2}$ *mile*.

73. $u=\text{cosec}\,4\theta\{(\cot\theta+\cot 2\theta)\sin 4\theta+\cos 4\theta\}$

$$=\text{cosec}\,4\theta\left\{\frac{\sin(2\theta+\theta)}{\sin\theta\sin 2\theta}\sin 4\theta+\cos 4\theta\right\}$$

$$=\text{cosec}\,4\theta\left\{\frac{\sin 3\theta}{\sin\theta}\times 2\cos 2\theta+\cos 4\theta\right\}$$

$=\operatorname{cosec}4\theta\{(3-4\sin^2\theta)2\cos2\theta+\cos4\theta\}$
$=\operatorname{cosec}4\theta\{(1+2\cos2\theta)2\cos2\theta+\cos4\theta\}$
$=\operatorname{cosec}4\theta\{2\cos2\theta+2(1+\cos4\theta)+\cos4\theta\}=$ &c.

174. By (93, Ex. 4) $d^2=2rh$, (where h is expressed in *miles*,)
$=8000\times\frac{h'}{1760\times3}$, (if h' be the height in *feet*,)
$=\frac{100}{66}h'=\frac{3}{2}h'$, nearly.

175. By (122) $\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{1}{2}C=$ (in this case)

$$\tfrac{1}{8}\cot32°\,6'=\tfrac{1}{8}\tan57°\,54';$$

$\therefore L\tan\frac{1}{2}(A-B)=L\tan57°\,54'-3\log2=9.2994355$:
here then $D=6588$, $d=1139$, $n=\frac{1139}{6588}\times60=10$;
and $\frac{1}{2}(A-B)=11°\,16'\,10''$, or $A-B=22°\,32'\,20''$;
also $A+B=180°-C=115°\,48'$;

$$\therefore A=69°\,10'\,10'',\ B=46°\,37'\,50''.$$

176. $u=\frac{1}{4}\sin\theta(\sin2\theta)^2=\frac{1}{4}\sin\theta\times\frac{1}{2}(1-\cos4\theta)$
$=\frac{1}{8}\sin\theta-\frac{1}{8}\sin\theta\cos4\theta=\frac{1}{8}\sin\theta-\frac{1}{16}\{\sin(4\theta+\theta)-\sin(4\theta-\theta)\}$.

177. Regarding the diameter of the disc as a small circular arc to radius r, (r being its distance from the eye,) $\therefore$ (since $a=r\theta$) we have $6=r\theta$, where $\theta=$ circ. meas. of $0°.95$ or $3420''$, which is $\frac{3420}{206265}$; $\therefore r=6\times\frac{206265}{3420}$ *inches* $=30$ *feet*.

178. $\frac{1}{2}(\log2+L\sec A-10)=-\log5+\frac{1}{3}\{2(L\cos B-10)-(L\sin C-10)\}$; whence we get $3L\sec A-4L\cos B+2L\sin C=10-6\log5-3\log2=4+3\log2=4.90309$.

179. $u=(1+\cot^2\theta)+(1+\cot^2\phi)-2\frac{\cos\theta\cos\phi+\sin\theta\sin\phi}{\sin\theta\sin\phi}$
$=2+\cot^2\theta+\cot^2\phi-2\cot\theta\cot\phi-2=(\cot\theta\backsim\cot\phi)^2$.

180. Let $ABCD$ be the field, $AB=30$, $BC=60$, $CD=110$, in *yards*; draw AE parallel to BC; then $AD^2=AE^2+DE^2$ $=60^2+80^2=10000$, and $AD=100$: also area (127.2) $=\frac{1}{2}(AB+CD)BC=4200$ *sq. yds.*

181. Let O be the centre of the inscribed circle: then $\angle EDF$ $=\frac{1}{2}\angle BOC=\frac{1}{2}\{\pi-\frac{1}{2}B-\frac{1}{2}C\}=\frac{1}{2}\{\pi-\frac{1}{2}(\pi-A)\}$ $=\frac{1}{4}(\pi+A)$.

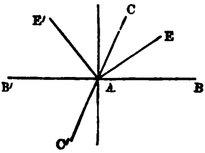
182. $2c=2\cos(\phi-\theta+a)\cos(\theta-a)=\cos\phi+\cos(\phi-2\theta+2a)$ $=$ also $\cos\phi+\cos(\phi-2\theta-2a)$ from the other equations: $\therefore \phi-2\theta+2a=\pm(\phi-2\theta-2a)$, where the upper sign gives no solution, but the lower gives $2\theta=\phi$:
hence $2c=\cos\phi+\cos2a$, from the first equation.

183. *Any* number m must be of the form $6n$ or $6n\pm1$ or $6n\pm2$,

or $6n+3$; $\therefore \sec\frac{1}{3}m\pi = \sec 2n\pi = 1$, or $= \sec(2n\pi \pm \frac{1}{3}\pi)$ $= \sec\frac{1}{3}\pi = 2$, or $= \sec(2n\pi \pm \frac{2}{3}\pi) = \sec\frac{2}{3}\pi = -2$, or $= \sec(2n\pi + \pi) = \sec\pi = -1$.

84. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A} = \dfrac{2x}{1-x^2}$ suppose;

$\therefore x^2 - 2x\cot 2A - 1 = 0$, and $x = -\cot 2A \pm \sqrt{(1+\cot^2 2A)}$ $=$ &c. Now this expression also $= -\cot 2A \pm \operatorname{cosec} 2A$ $= -\dfrac{\cos 2A \pm 1}{\sin 2A} = -\cot A$ or $+\tan A$, by (74).



Let $\angle BAC = 2A$; produce CA to C'; then $\tan 2A = \tan BAC$ or $\tan BAC'$, and if we bisect these angles by AE, AE', then $\tan BAE = +\tan A$, $\tan BAE'$ $= \tan\frac{1}{2}(180° + 2A) = \tan(90° + A)$ $= -\cot A$.

85. Here $2^{3x}5^{6-3x} = 2^{4x+3}5^{x}$; $\therefore 3x\log 2 + (6-3x)(1-\log 2)$ $= (4x+3)\log 2 + x(1-\log 2)$, whence $(3x-9)\log 2 = 4x-6$, or $x = \dfrac{6-9\log 2}{4-3\log 2} = \dfrac{3.29073}{3.09691} = 1.06$.

86. Since AE is bisected in D, we have, by a well-known property (see Key to Colenso's Geom. Prob. 2. 33. or 211, below),

$AC^2 + CE^2 = 2(DE^2 + CD^2)$, or

$b^2 = 2DE^2 + 2CD^2 - CE^2$, and, similarly,

$a^2 = 2DE^2 + 2CE^2 - CD^2$;

$\therefore c^2 = a^2 + b^2 = 4DE^2 + CD^2 + CE^2$, and $CD^2 + CE^2 + DE^2$ $= c^2 - 3DE^2 = c^2 - \frac{1}{3}c^2 = \frac{2}{3}c^2$.

87. Let $\dfrac{a(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + b\sin\theta}{a'(\sin\theta\cos\alpha + \cos\theta\sin\alpha) + b'\cos\theta} = m$, for all values of θ;

then $\{a\cos\alpha - m(a'\sin\alpha + b')\}\cos\theta = (ma'\cos\alpha + a\sin\alpha - b)\sin\theta$, or $a\cos\alpha - m(a'\sin\alpha + b') + (b - a\sin\alpha - ma'\cos\alpha)\tan\theta = 0$, for all values of θ: hence

(1) $a\cos\alpha - m(a'\sin\alpha + b') = 0$, (2) $b - a\sin\alpha - ma'\cos\alpha = 0$;

from (1) $m = \dfrac{a\cos\alpha}{a'\sin\alpha + b'} = \dfrac{b - a\sin\alpha}{a'\cos\alpha}$ from (2);

$\therefore aa'\cos^2\alpha = a'b\sin\alpha + bb' - aa'\sin^2\alpha - ab'\sin\alpha$,

or $aa' - bb' = (a'b - ab')\sin\alpha$.

88. Given $\tan\frac{1}{2}A + \tan\frac{1}{2}C = 2\tan\frac{1}{2}B$,

or $\tan\frac{1}{2}A - \tan\frac{1}{2}B = \tan\frac{1}{2}B - \tan\frac{1}{2}C$;

$\therefore \dfrac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}A\cos\frac{1}{2}B} = \dfrac{\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$

whence $2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}C = 2\sin\frac{1}{2}(B-C)\cos\frac{1}{2}A$,
that is, $\sin\frac{1}{2}(A-B+C)+\sin\frac{1}{2}(A-B-C)$
$=\sin\frac{1}{2}(B-C+A)+\sin\frac{1}{2}(B-C-A)$,
or $\cos B-\cos A=\cos C-\cos B$, since $\sin\frac{1}{2}(A-B+C)$
$=\sin\frac{1}{2}(180^\circ-2B)=\sin(90^\circ-B)=\cos B$, &c.

189. $\sin\theta=\theta-\frac{1}{6}\theta^3$ nearly, or $\frac{\sin\theta}{\theta}=1-\frac{1}{6}\theta^2$;
$\therefore\ \theta^2=6\left(1-\frac{2165}{2166}\right)=\frac{1}{361}$, and $\theta=\frac{1}{19}$;
hence $\theta_{\circ}$ contains $\frac{1}{19}\times 57^\circ.29577=3^\circ$ nearly.

190. $D=4130$; $\therefore$ (i) $d=\frac{12}{60}\times 4130=826$,
and $L\sin 17^\circ\,12''=9.4660179$; (ii) $d=541$:
$\therefore\ x''=\frac{541}{4130}\times 60''=7''.8=8''$, nearly.

191. $u=\{a\cos(x+2)\theta+b\sin(x+2)\theta\}-2\cos\theta\times$
$\{a\cos(x+1)\theta+b\sin(x+1)\theta\}+\{a\cos x\theta+b\sin x\theta\}$
$=a\{\cos(x+2)\theta+\cos x\theta-2\cos(x+1)\theta\cos\theta\}$
$+b\{\sin(x+2)\theta+\sin x\theta-2\sin(x+1)\theta\cos\theta\}=0$.
since $\cos(x+2)\theta+\cos x\theta=2\cos(x+1)\theta\cos\theta$, &c.

192. $\frac{\sin\phi}{\tan\phi}=\cos\phi=m\tan\theta\div\frac{\sin\theta}{n}=mn\sec\theta$;
$\therefore\ \sin^2\phi+\cos^2\phi=1=m^2\tan^2\theta+m^2n^2\sec^2\theta$
$=m^2\sec^2\theta-m^2+m^2n^2\sec^2\theta$,
or $(m^2+m^2n^2)\sec^2\theta=m^2+1$, &c.

193. Multiply each term by $2\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$;
then $2\sin\alpha\sin(\gamma-\beta)+2\sin\beta\sin(\alpha-\gamma)+2\sin\gamma\sin(\beta-\alpha)$
$=\{\cos(\alpha+\beta-\gamma)-\cos(\alpha+\gamma-\beta)\}$
$+\{\cos(\beta+\gamma-\alpha)-\cos(\beta+\alpha-\gamma)\}$
$+\{\cos(\gamma+\alpha-\beta)-\cos(\gamma+\beta-\alpha)\}=0$.

194. $r\operatorname{vers}A=6\,in.$, $r\operatorname{chd}A=36\,in.=6r\operatorname{vers}A$;
$\therefore\ 2\sin\frac{1}{2}A=6(1-\cos A)=12\sin^2\frac{1}{2}A$;
whence $\sin\frac{1}{2}A=\frac{1}{6}$, and $\operatorname{chd}\frac{1}{2}A=\frac{1}{3}$;
$\therefore\ \frac{1}{3}r=36\,in.=1\,yd.$, and $r=3\,yds.$

195. $BD=c\frac{\sin BAD}{\sin BDA}=CD=b\frac{\sin CAD}{\sin CDA}$;
$\therefore\ c\sin BAD=b\sin CAD$, or $\sin C\sin\theta=\sin B\sin\phi$,
suppose, if $\angle BAD=\theta$, $\angle CAD=\phi$:
also $A=\theta+\phi=180^\circ-B-C$; $\therefore\ \theta+C=180^\circ-B-\phi$;
$\therefore\ \sin(\theta+C)=\sin(B+\phi)$,

and $\therefore \dfrac{\sin(\theta+C)}{\sin C\sin\theta}=\dfrac{\sin(B+\phi)}{\sin B\sin\phi}$,

whence, expanding, $\cot C+\cot\theta=\cot\phi+\cot B$.

96. (1) D being negative for the *cosine*, (103 Ex. 2. N.B) $=-491$,
and $d=+383$; $\therefore n=-\frac{383}{491}\times 60''=-46.8$,
and $A=21^\circ 15'-46''.8=21^\circ 14' 13''$, nearly:
(2) $L\operatorname{cosec} 68^\circ 45' 24''=L\sec 21^\circ 14' 36''$
$=20-L\cos 21^\circ 14' 36''$ by (155);
now $21^\circ 14' 36''=21^\circ 15'-24''$, so that $n=-24$,
and $d=-\frac{24}{60}\times -491=196.4$;
$\therefore L\cos 21^\circ 14' 36''=9.96943924$,
and $L\sec 21^\circ 14' 36''=10.0305608$, nearly.

97. $\tan\theta+\dfrac{1}{\tan\theta}=\dfrac{2(m^2+n^2)}{m^2-n^2}$,

or $(m^2-n^2)\tan^2\theta-2(m^2+n^2)\tan\theta+(m^2-n^2)=0$,

$$\therefore \tan\theta=\frac{m^2+n^2\pm 2mn}{m^2-n^2}=\frac{m+n}{m-n}, \text{ or } \frac{m-n}{m+n};$$

and $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$ by (76) $=\dfrac{(m+n)^2 \backsim (m-n)^2}{(m+n)^2+(m-n)^2}$

$=\pm\dfrac{2mn}{m^2+n^2}$.

98. $\cos(A-C)\cos B=\cos(A+C-B)$
$=\cos(A+C)\cos B+\sin(A+C)\sin B$;
$\therefore \{\cos(A-C)-\cos(A+C)\}\cos B=2\sin A\sin C\cos B$
$=\sin(A+C)\sin B$,

or $2\cot B=\dfrac{\sin(A+C)}{\sin A\sin C}=\cot A+\cot C$;

$\therefore \cot A$, $\cot B$, $\cot C$, are in $A.P$, and $\tan A$, $\tan B$, $\tan C$, in $H\cdot P$.

99. Here $(\cos A+\sin A)^2=1+\sin 2A=2$, or $\sin 2A=1$;
$\therefore 2A=90^\circ$, or $n\pi+(-1)^n\frac{1}{2}\pi$.

00. $A=32^\circ 15'$; let B be the other angle: then there will be two solutions, according as we take (1) $a=468$, $c=320$, or (2) $a=320$, $c=468$:

(1) $\sin C=\dfrac{c}{a}\sin A=\dfrac{80}{117}\sin A$;

$\therefore L\sin C = \log 80 - \log 117 + L\sin A$
$= (3\log 2 + 1) - (2\log 3 + \log 13) + (20 - L\operatorname{cosec} 32^\circ 15')$
$= 9.5621316 = L\sin 21^\circ 23'$;
$\therefore B = 180^\circ - 32^\circ 15' - 21^\circ 23' = 126^\circ 22'$:

(2) $\sin C = \frac{117}{80}\sin A$;

$\therefore L\sin C =$ (as before) $9.8923236 = L\sin 51^\circ 18'$;
$\therefore B = 180^\circ - 32^\circ 15' - 51^\circ 18' = 96^\circ 27'$.

201. $u = (\cos\theta + \cos 3\theta) + \cos 2\theta = 2\cos\theta\cos 2\theta + \cos 2\theta = 0$;
$\therefore$ (i) $\cos 2\theta = 0$, or $2\theta = 90^\circ$, or $2n\pi \pm \frac{1}{2}\pi$,
and (ii) $\cos\theta = -\frac{1}{2}$, or $\theta = 120^\circ$, or $2n\pi \pm \frac{2}{3}\pi$.

202. Take O the centre of inscribed circle, OD, OE, OF, the perpendiculars on sides; then AEF, BDF, CDE, DEF, are the four triangles.

203. $A' = BOC = 180^\circ - \frac{1}{2}B - \frac{1}{2}C = 90^\circ + \frac{1}{2}A$, &c.:
$\therefore u = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$
$= 2\cos\frac{1}{2}A\{\cos(\frac{1}{2}B + \frac{1}{2}C) + \cos(\frac{1}{2}B - \frac{1}{2}C)\}$
$= 2\cos\frac{1}{2}A\sin\frac{1}{2}A + 2\sin\frac{1}{2}(B + C)\cos\frac{1}{2}(B - C) =$ &c.

204. Height $= 120\tan 60^\circ 14''$:
now $\tan 60^\circ = \sqrt{3} = 1.7320508$, and $\tan 60^\circ 1' = 1.7332149$;
$\therefore D = 11641$, and d (for $14''$) $= \frac{14}{60} \times 11641 = 2716$;
$\therefore \tan 60^\circ 14'' = 1.7323224$, and height $= 208$ *ft.*, nearly.

205. $\cos 2A = \sin A = \cos(90^\circ - A)$;
$\therefore 2A = 90^\circ - A$, or (generally) $= 2n\pi \pm (\frac{1}{2}\pi - A)$;
$\therefore A = 30^\circ$, or (generally) $=$ (i) $(4n + 1)\frac{1}{6}\pi$,
or (ii) $(4n - 1)\frac{1}{2}\pi$.

206. $u = (86)\tan^{-1}\dfrac{\tan A + \cot 2A}{1 - \tan A\cot 2A}$

$$= \tan^{-1}\frac{\sin A\sin 2A + \cos A\cos 2A}{\sin 2A\cos A - \cos 2A\sin A}$$

$$= \tan^{-1}\frac{\cos(2A - A)}{\sin(2A - A)} = \text{\&c.}$$

207. Since $\dfrac{AF}{BF} = \dfrac{1}{n-1}$,

$$\therefore \frac{AF}{AF + BF} \text{ or } \frac{AF}{AB} = \frac{1}{n}, \text{ and } \frac{BF}{AB} = \frac{n-1}{n}, \text{ \&c.}$$

hence $\dfrac{\text{area } AEF}{\text{area } ABC} = \dfrac{\frac{1}{2}AE.AF\sin A}{\frac{1}{2}AB.AC\sin A} = \dfrac{AE}{AC} \times \dfrac{AF}{AB}$

$$= \frac{n-1}{n} \times \frac{1}{n} = \frac{n-1}{n^2}, \text{ \&c.};$$

$$\therefore \frac{\text{area}\,DEF}{\text{area}\,ABC}=\frac{ABC-(AEF+BDF+CDE)}{ABC}$$

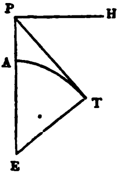
$$=1-\frac{3(n-1)}{n^2}.$$

08. Given $\angle NAB=\phi$, $\angle CAB=\alpha$, $\angle CBA=\beta$, $\angle DAC=\theta$, D being the top of the pillar CD:

then height $=AC\tan\theta=a\dfrac{\sin\beta}{\sin(\alpha+\beta)}\tan\theta$,

and $\angle NAC=\phi-\alpha$.

09. P the top of the peak, PT a tangent to the earth's surface, so that T is a point in the horizon, E the earth's centre, $\angle HPT$, the dip of horizon, $=1°\ 58'$ $=\angle AET$.



Now $EP=ET\sec E=AP+ET$;

$\therefore AP=r(\sec E-1)=4000\times .0005894$
$=2$ *miles* 629 *yds.*

10. By the Table in (137) we get
$\log 55=\log 11+(1-\log 2)=1.7403627$,
$\log 40=1+2\log 2=1.6020600$;
$\therefore L\sin B=L\sin A+\log b-\log a=9.7705700$;
now, from the question, D (for $60''$) $=1731$,
and (for $L\sin B$) $d=1368$;
$\therefore n=\frac{1368}{1731}\times 60=47$; $\therefore B=36°\,7'\,47''$.

11. $AC^2=c^2+l^2-2cl\cos ADC$,
$BC^2=c^2+l^2+2cl\cos ADC$; $\therefore AC^2+BC^2=2(c^2+l^2)$,

and $\therefore \cos C=\dfrac{AC^2+BC^2-AB^2}{2AC.BC}=\dfrac{l^2-c^2}{AC.BC}$;

hence area $=\frac{1}{2}AC.BC\sin C=\frac{1}{2}(l^2-c^2)\tan C$.

12. By (85) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$,
or $(1+2+3)\tan A=1.2.3\tan^3 A$;
$\therefore \tan A=1$, $\tan B=2$, $\tan C=3$;
and $pqr=c\sin B.a\sin C.b\sin A$

$$=abc\times\frac{2}{\sqrt{5}}\times\frac{3}{\sqrt{10}}\times\frac{1}{\sqrt{2}}=\frac{3}{5}abc.$$

13. $u=2\cos(n-1)A\cos A=\cos A$;
$\therefore$ (i) $\cos A=0$, or $A=(2n+1)\frac{1}{2}\pi$,
(ii) $\cos(n-1)A=\frac{1}{2}$, or $(n-1)A=2n\pi\pm\frac{1}{3}\pi$.

214. $u=2+\{\cos^2\varsigma-\sin^2(\varsigma-\theta)\}+\{\cos^2(\varsigma-\phi)-\sin^2(\varsigma-\psi)\}$
$=$ by (84) $2+\cos(2\varsigma-\theta)\cos\theta+\cos(2\varsigma-\phi-\psi)\cos(\phi-\psi)$
$=2+\cos(\phi+\psi)\cos\theta+\cos\theta\cos(\phi-\psi)$
$=2+\cos\theta\{\cos(\phi+\psi)+\cos(\phi-\psi)\}=$ &c.

215. $\tan 36^\circ=\sqrt{\frac{1-\cos 72^\circ}{1+\cos 72^\circ}}=\sqrt{\frac{1-\sin 18^\circ}{1+\sin 18^\circ}}=\sqrt{\frac{5-\sqrt5}{3+\sqrt5}}$

$$=\frac{\sqrt{\{(5-\sqrt5)(3-\sqrt5)\}}}{\sqrt{(9-5)}}=\sqrt{(5-2\sqrt5)}:$$

$\tan 37\tfrac12^\circ=\sqrt{\frac{1-\cos 75^\circ}{1+\cos 75^\circ}}=\sqrt{\frac{2\sqrt2-\sqrt3+1}{2\sqrt2+\sqrt3-1}}$, by (71 Ex. 1).

216. Put $a=k\cos\phi$, $b=k\sin\phi$;

$\therefore\ \tan\phi=\frac{b}{a}$, which determines ϕ, and $k=a\sec\phi$:

then $u=k(\cos\phi\cos\theta\pm\sin\phi\sin\theta)=k\cos(\phi\mp\theta)$.

217. $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{25+36-16}{60}=\frac{3}{4}$,

and so $\cos B=\frac{9}{16}=\cos^2 A$:

also $L\cos B=10+\log\frac{9}{16}=10+2\log 3-4\log 2=9.7501226$;

$\therefore D$ (from question) $=-1857$, and d (for $\cos B$) $=-497$;

$\therefore n=\frac{497}{1857}\times 60=16$, and $B=55^\circ\,46'\,16''$.

218. OAB the quadrant, $OA=a$, $AB=c$, OD the bisecting radius, in which take C the centre of inscribed circle; now $CE(=a')\perp OA$; then $OD=OC+CD$, or $a=a'\sqrt2+a'$, and $a'=a\div(\sqrt2+1)=a(\sqrt2-1)$; $\therefore a+a'=a\sqrt2=c$.

219. Let a, a', be the sides of the two figures; then $3a=6a'$, and $a=2r\sin 60^\circ$, $a'=2r'\sin 30^\circ$;

$\therefore\ \frac{r}{r'}=\frac{a\sin 30^\circ}{a'\sin 60^\circ}=\frac{2}{\sqrt3}$, and the circles are as $r^2:r'^2=4:3$.

220. $\text{Area}=\frac12 ab\sin C=\frac12\cdot\frac{a^2+b^2-c^2}{2\cos C}\sin C$

$$=\frac{a^2+b^2-c^2}{4\tan(90^\circ-C)}=\text{\&c.}$$

221. Put $\cot A=k\cos\phi$, $\cos b=k\sin\phi$;

$\therefore \tan\phi=\cos b\tan A$, and $k=\cos b\operatorname{cosec}\phi$;

then $k(\cos\phi\sin C+\sin\phi\cos C)=k\sin(C+\phi)=\cot a\sin b$.

22. $\sin 4\theta + 2(\sin 4\theta + \sin 2\theta) = 0$,
or $2\sin 2\theta = -3\sin 4\theta = -6\sin 2\theta\cos 2\theta$;
$\therefore$ (i) $\sin 2\theta = 0$, or $2\theta = n\pi$, and $\tan\theta = \tan\frac{1}{2}n\pi = 0$ or ∞;

$$\text{(ii)}\ \cos 2\theta = -\frac{1}{3},\ \text{and}\ \tan\theta = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{2}.$$

23. $$\frac{AD}{BD} = (\text{Euc. vi. 3.})\ \frac{AC}{BC} = \tan B = \sqrt{\frac{1-\cos 2B}{1+\cos 2B}}$$
$$= \sqrt{\frac{1-\sin(90^\circ - 2B)}{1+\sin(90^\circ - 2B)}} = \sqrt{\frac{1-\sin(A-B)}{1+\sin(A-B)}}$$
$$= \text{by (73)}\ \frac{\cos\frac{1}{2}(A-B) - \sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A-B) + \sin\frac{1}{2}(A-B)} = \&\text{c.}$$

24. $2 - 4\sin^2 A = 2\sin A + 1$; $\therefore$ $4\sin^2 A + 2\sin A - 1 = 0$,
or $\sin A = \frac{1}{4}(-1 \pm \sqrt{5}) = \sin 18^\circ$, or $\sin(-54^\circ)$:
hence $A = n\pi + (-1)^n \frac{1}{10}\pi$, or $n\pi - (-1)^n \frac{3}{10}\pi$.

25. In triangle ABC we have given $\angle ABC = 30^\circ$,
$\angle BCA = 135^\circ$, $BC = 1$ *mile*;

$$\therefore AB = BC\frac{\sin 135^\circ}{\sin 15^\circ} = \frac{1}{\sqrt{2}} + \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{2}{\sqrt{3}-1} = \sqrt{3}+1;$$

and vertical height of mountain $= AB\sin 60^\circ = \frac{1}{2}(3+\sqrt{3})$
$= 2.3660$ *miles* $= 12492$ *ft.*

26. Squaring the given equation, we get
$\sec^2\alpha(\tan^2\theta + 1) + 2\sec\alpha\tan\alpha\sec\theta\tan\theta + \tan^2\alpha(\sec^2\theta - 1)$
$= \sec^2\beta$, or $(\sec\alpha\tan\theta + \tan\alpha\sec\theta)^2 = \sec^2\beta - \sec^2\alpha + \tan^2\alpha$
$= \sec^2\beta - 1 = \tan^2\beta$.

27. $$u = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos A + \frac{1}{\sqrt{2}}\sin A\right)$$
$$+ \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos B + \frac{1}{\sqrt{2}}\sin B\right)$$
$$= \sqrt{2}\{\cos(45^\circ - A) + \cos(45^\circ - B)\}$$
$$= \sqrt{2}\times 2\cos\tfrac{1}{2}(45^\circ - A + 45^\circ - B)\cos\tfrac{1}{2}(45^\circ - A - 45^\circ + B).$$

28. $u = (\cos^4\frac{1}{8}\pi + \cos^4\frac{7}{8}\pi) + (\cos^4\frac{3}{8}\pi + \cos^4\frac{5}{8}\pi)$
$= 2\cos^4\frac{1}{8}\pi + 2\cos^4\frac{3}{8}\pi = 2(\cos^4\frac{1}{8}\pi + \sin^4\frac{1}{8}\pi)$
$= 2\{(\cos^2\frac{1}{8}\pi + \sin^2\frac{1}{8}\pi)^2 - 2\sin^2\frac{1}{8}\pi\cos^2\frac{1}{8}\pi\}$
$= 2\{1 - \frac{1}{2}\sin^2\frac{1}{4}\pi\} = 2\{1 - \frac{1}{4}\} = \frac{3}{2}$.

29. A, B, C, the stations in order, DE the steeple; $AB = a$,
$BC = b$, $\angle DBE = 90^\circ - A$, $\angle DCE = 2A$, $DE = x$;
then $a = AE - BE = x(\cot A - \tan A) = 2x\cot 2A = 2CE$;

and $\angle ADC = \angle(DCE - DAE) = A = \angle DAC$;

$\therefore CD = CA = a + b$,

and $x = \sqrt{\{(a+b)^2 - CE^2\}} = \sqrt{\{(a+b)^2 - \frac{1}{4}a^2\}}$.

230. $u = \sqrt{(1 - \cos\theta)} = \dfrac{\sin\theta}{\sqrt{(1+\cos\theta)}}$

$= \dfrac{\sin\theta}{\sqrt{\{1 + \sin(90^\circ - \theta)\}}} =$ &c. by (73).

231. $u = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) + \sin\frac{1}{2}(A+B)$

$= 2\cos\frac{1}{2}(A+B)\,\{\cos\frac{1}{2}(A-B) + \sin\frac{1}{2}(A+B)\}$

$= 2\cos\frac{1}{2}(A+B)\{\cos\frac{1}{2}(A-B) + \cos\{90^\circ - \frac{1}{2}(A+B)\}\}$

$= 2\cos\frac{1}{2}(A+B)\{2\cos\frac{1}{2}(90^\circ - \frac{1}{2}B)\cos\frac{1}{2}(90 - \frac{1}{2}A)\} =$ &c.

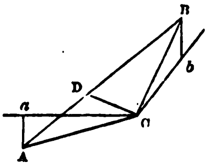
232. $2\tan^{-1}\frac{1}{5} = \tan^{-1}\dfrac{\frac{2}{5}}{1 - \frac{1}{25}} = \tan^{-1}\frac{5}{12}$,

$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{5}{12} = \tan^{-1}\dfrac{\frac{1}{4} + \frac{5}{12}}{1 - \frac{5}{48}} = \tan^{-1}\frac{32}{43}$,

$\tan^{-1}\frac{32}{43} + \tan^{-1}\frac{1}{6} = \tan^{-1}\dfrac{\frac{32}{43} + \frac{1}{6}}{1 - \frac{32}{258}} = \tan^{-1}\frac{235}{226}$

$=$ (by question) $\frac{1}{2}\pi - \cot^{-1}x = \tan^{-1}x$; $\therefore x = \frac{235}{226}$.

233.



$BC = BD\dfrac{\sin BDC}{\sin BCD} = a\dfrac{\sin\phi}{\sin(\theta+\phi)}$;

$\therefore Bb = a\dfrac{\sin\phi\sin\beta}{\sin(\theta+\phi)}$;

$AB = BC\dfrac{\sin\psi}{\sin(\theta+\psi)}$

$= a\dfrac{\sin\phi\sin\psi}{\sin(\theta+\phi)\sin(\theta+\psi)}$;

$\therefore AC = AB\dfrac{\sin\theta}{\sin\psi} = a\dfrac{\sin\theta\sin\phi}{\sin(\theta+\phi)\sin(\theta+\psi)}$,

and $Aa = AC\sin\alpha$;

then $Aa + Bb =$ *Ans.*

234. $m^2 = c^2 + l^2 - 2cl\cos\frac{1}{2}A$, $n^2 = b^2 + l^2 - 2bl\cos\frac{1}{2}A$;

$\therefore bm^2 - cn^2 = bc^2 - cb^2 + l^2(b - c)$,

or, $\dfrac{m^2}{c} - \dfrac{n^2}{b} = c - b + l^2\left(\dfrac{1}{c} - \dfrac{1}{b}\right)$;

so $\dfrac{n^2}{a} - \dfrac{l^2}{c} = a - c + m^2\left(\dfrac{1}{a} - \dfrac{1}{c}\right)$, and $\dfrac{l^2}{b}-\dfrac{m^2}{a}=b-a+n^2\left(\dfrac{1}{b}-\dfrac{1}{a}\right)$;

adding these three equations, we get the required result.

35. $u=$ by (73) $\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta=(1-2\sin^2\frac{1}{4}\theta)+2\sin\frac{1}{4}\theta\cos\frac{1}{4}\theta$
$=1+2\sin\frac{1}{4}\theta(\cos\frac{1}{4}\theta-\sin\frac{1}{4}\theta)=$ &c. by (73).

36. $u=2\cos(x+1)a.\cos\frac{1}{2}a=\sin a=2\sin\frac{1}{2}a\cos\frac{1}{2}a$;
$\therefore\ \cos(x+1)a=\sin\frac{1}{2}a=\cos(\frac{1}{2}\pi-\frac{1}{2}a)$,
and $(x+1)a=2n\pi\pm\frac{1}{2}(\pi-a)$.

37. Here $\sin A-\sin B=\sin B-\sin C$,
or $\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)=\sin\frac{1}{2}(B-C)\cos\frac{1}{2}(B+C)$;
$\therefore\ \sin\frac{1}{2}(A-B)\sin\frac{1}{2}C=\sin\frac{1}{2}(B-C)\sin\frac{1}{2}A$,
or $\sin\frac{1}{2}(A-B)\div\sin\frac{1}{2}A\sin\frac{1}{2}B$
$=\sin\frac{1}{2}(B-C)\div\sin\frac{1}{2}B\sin\frac{1}{2}C$,
that is, $\cot\frac{1}{2}B-\cot\frac{1}{2}A=\cot\frac{1}{2}C-\cot\frac{1}{2}B$,
whence $\cot\frac{1}{2}A$, $\cot\frac{1}{2}B$, $\cot\frac{1}{2}C$, are in $A.P.$

38. Let $\theta=\cos^{-1}\sqrt{\frac{2}{3}}$; then $\sin\theta=\sqrt{\frac{1}{3}}$:

let $\phi=\cos^{-1}\dfrac{\sqrt{6}+1}{2\sqrt{3}}$; then $\sin\phi=\sqrt{\dfrac{5-2\sqrt{6}}{12}}=\dfrac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}$;

$$\therefore\ \sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$$
$$=\sqrt{\frac{1}{3}}\cdot\frac{\sqrt{6}+1}{2\sqrt{3}}-\sqrt{\frac{2}{3}}\cdot\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}=\frac{\sqrt{6}+1}{6}-\frac{\sqrt{6}-2}{6}$$
$$=\frac{1}{2}=\sin\frac{\pi}{6}.$$

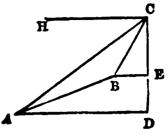
39. $\tan\frac{1}{2}B\tan\frac{1}{2}C=\sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}\times\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}=\dfrac{s-a}{s}$,

$$\therefore\ a=s(1-\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C)=s\cos\tfrac{1}{2}(B+C)\div\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$$
$$=s\sin\tfrac{1}{2}A\sec\tfrac{1}{2}B\sec\tfrac{1}{2}C.$$

40. $\tan(\phi-\theta)=\dfrac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\dfrac{\tan\phi-\cos\alpha\tan\phi}{1+\cos\alpha\tan^2\phi}$

$$=\frac{(1-\cos\alpha)\sin\phi\cos\phi}{\cos^2\phi+\cos\alpha\sin^2\phi}=\frac{\sin^2\frac{1}{2}\alpha\sin2\phi}{\frac{1}{2}(1+\cos2\phi)+\frac{1}{2}(1-\cos2\phi)\cos\alpha}$$
$$=\frac{\sin^2\frac{1}{2}\alpha\sin2\phi}{\cos^2\frac{1}{2}\alpha+\sin^2\frac{1}{2}\alpha\cos2\phi}=\text{\&c.}$$

41.



$\angle BAD=\alpha$, $\angle CBE=\beta$, $CD=m$,
$\angle HCA=\angle CAD=\gamma$:
then $m=AB\sin\alpha+BC\sin\beta$,
$m\cot\gamma=AD=AB\cos\alpha+BC\cos\beta$;
$\therefore\ AB=\dfrac{m\cot\gamma\sin\beta-m\cos\beta}{\sin(\beta-\alpha)}$

$$=\frac{m\sin(\beta-\gamma)}{\sin\gamma\sin(\beta-\alpha)},\quad BC=\frac{m\sin(\gamma-\alpha)}{\sin\gamma\sin(\beta-\alpha)};$$

$$\therefore AB+BC=m\frac{\sin(\beta-\gamma)+\sin(\gamma-\alpha)}{\sin\gamma\sin(\beta-\alpha)}$$

$$=m\frac{2\sin\frac{1}{2}(\beta-\alpha)\cos\{\frac{1}{2}(\beta+\alpha)-\gamma\}}{2\sin\gamma\sin\frac{1}{2}(\beta-\alpha)\cos\frac{1}{2}(\beta-\alpha)}=\&c.$$

242. Let CD, the line bisecting c in D, $=l$;
then $b^2=l^2+\frac{1}{4}c^2-lc\cos ADC$, $a^2=l^2+\frac{1}{4}c^2+lc\cos ADC$;
$\therefore a^2+b^2=2l^2+\frac{1}{2}c^2$, and $l^2=\frac{1}{4}\{2(a^2+b^2)-c^2\}$.

243. $2\sin^2 x=\sec x-\cos x=\sin^2 x\div\cos x$;
$\therefore$ (i) $\sin x=0$, and $x=0$ or $n\pi$;
(ii) $\cos x=\frac{1}{2}$, and $x=60°$ or $2n\pi\pm\frac{1}{3}\pi$.

244. $$u=\frac{\cos^2 x+\operatorname{cosec}^2\alpha\sin^2 x}{\cos^2 x+\operatorname{cosec}^2\beta\sin^2 x}=\frac{1+(\operatorname{cosec}^2\alpha-1)\sin^2 x}{1+(\operatorname{cosec}^2\beta-1)\sin^2 x}=\&c.$$

245. $$\cos^{-1}\frac{1-m^2}{1+m^2}=\sec^{-1}\frac{1+m^2}{1-m^2}=\tan^{-1}\sqrt{(\sec^2-1)}$$

$$=\tan^{-1}\sqrt{\left\{\left(\frac{1+m^2}{1-m^2}\right)^2-1\right\}}=\tan^{-1}\frac{2m}{1-m^2};$$

hence $u=\tan^{-1}\dfrac{2m}{1-m^2}-\tan^{-1}\dfrac{2n}{1-n^2}=\&c.$, by (86).

246. Let P be the point; then, each of the angles at P being 120°,
we have $\cos APB=-\frac{1}{2}=\dfrac{\alpha^2+\beta^2-c^2}{2\alpha\beta}$,

$$\text{or } c=\sqrt{(\alpha^2+\alpha\beta+\beta^2)}:$$

also area $=$ area $APB+$ area $APC+$ area BPC
$=\frac{1}{2}\alpha\beta\sin 120°+\frac{1}{2}\alpha\gamma\sin 120°+\frac{1}{2}\beta\gamma\sin 120°=\&c.$

247. By (86) we get, after simplification of the complex fractions,

$$u=\tan^{-1}\frac{\sin\phi(1-2x\sin\phi+x^2)}{\cos\phi(1-2x\sin\phi+x^2)}=\tan^{-1}\tan\phi=\phi.$$

248. (i) $A=\pm 45°$, or, in general terms, $=n\pi\pm(-1)^n\frac{1}{4}\pi$, which may be written $n\pi\pm\frac{1}{4}\pi$, since it is useless, with the double sign, to retain the factor $(-1)^n$:
(ii) $A=60°$, and $A=120°$, or, in general terms, $A=2n\pi\pm\frac{1}{3}\pi$, and $A=2n\pi\pm\frac{2}{3}\pi$:
(iii) $\dfrac{1}{\sqrt{2}}\cos A+\dfrac{1}{\sqrt{2}}\sin A=\cos(A-45°)=1$;
$\therefore A-45°=0$, or $=2n\pi$; $\therefore A=45°$, or $2n\pi+\frac{1}{4}\pi$.

49. Diam. of small circle $=\frac{AD}{\sin C}=\frac{AD}{\sin 2A}=\frac{AD}{\sin 72^\circ}$,

since $\angle(BAD+ABD+ADB)=A+2A+2A=180^\circ$; hence the areas are as

$$AB^2 : \frac{AD^2}{4\sin^2 72^\circ}=4\cos^2 18^\circ : 1=5+\sqrt{5} : 2.$$

50. Here $\cos a\cos x-(\sin a+\cos a)\sin x=\sin b$,

or $k\cos(\phi+x)=\sin b$, if $k\cos\phi=\cos a$, $k\sin\phi=\sin a+\cos a$;

$$\therefore \cos(\phi+x)=\frac{\sin b}{k}=\frac{\sin b}{\cos a}\cos\phi,$$

$$\text{and } \frac{k\sin\phi}{k\cos\phi}=\tan\phi=\frac{\sin a+\cos a}{\cos a}=\frac{\frac{1}{\sqrt{2}}\sin a+\frac{1}{\sqrt{2}}\cos a}{\frac{1}{\sqrt{2}}\cos a}$$

$$=\frac{\sin(45^\circ+a)}{\cos 45^\circ\cos a}.$$

51. Here $\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}=4$, or $1=2\sin 2\theta$;

$\therefore \sin 2\theta=\frac{1}{2}=\sin 30^\circ$, and $2\theta=30^\circ$ or $n\pi+(-1)^n\sin\frac{1}{6}\pi$.

52. $u=\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+1}=$ by (86) $\tan^{-1}\frac{1}{n^2+n+1}$

$$=\cot^{-1}(1+n+n^2);$$

$$\therefore S=(\cot^{-1}1-\cot^{-1}2)+(\cot^{-1}2-\cot^{-1}3)+\&c.$$
$$+\{\cot^{-1}n-\cot^{-1}(n+1)\}$$
$$=\cot^{-1}1-\cot^{-1}(n+1)=\tan^{-1}1-\tan^{-1}\frac{1}{n+1}$$
$$=\tan^{-1}\frac{n}{n+2}.$$

53. $AC^2=AD^2+CD^2-2AD.CD\cos ADC$,
$AB^2=AD^2+BD^2-2AD.BD\cos ADB$;
$\therefore AC^2+AB^2=2(AD^2+CD^2)$,
since $BD=CD$, and $\cos ADB=-\cos ADC$:
hence $b^2+(a^2+b^2)=2m^2+\frac{1}{2}a^2$, or $4b^2+a^2=2m^2$;
so $4a^2+b^2=2n^2$;
hence $15a^2=2(4n^2-m^2)$, and $15b^2=2(4m^2-n^2)$;

$$\therefore \tan A=\frac{a}{b}=\sqrt{\frac{4n^2-m^2}{4m^2-n^2}}.$$

54. *By (75)* $u=\frac{4(\cos^3 A-\sin^3 A)-3(\cos A-\sin A)}{3(\cos A+\sin A)-4(\cos^3 A+\sin^3 A)}$

$$=\frac{4(\cos^2 A+\cos A\sin A+\sin^2 A)-3}{3-4(\cos^2 A-\cos A\sin A+\sin^2 A)}\times\frac{\cos A-\sin A}{\cos A+\sin A}$$

$$=\frac{4\sin A\cos A+1}{4\sin A\cos A-1}\times\frac{1-\tan A}{1+\tan A}=\&\text{c.}$$

255. Let A be the least angle;
then $A+B+C=A+2A+4A=7A=180^\circ$,
and $A=25\tfrac{5}{7}^\circ$, or $\tfrac{1}{2}A=12\tfrac{6}{7}^\circ=12^\circ 51' 25\tfrac{5}{7}''$:

$$\text{now } \frac{c}{a+b+c}=\frac{\sin C}{\sin A+\sin B+\sin C}$$

$$=\frac{\sin C}{4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}, \text{ as in (Ex. 15. 4)}$$

$$=\frac{\sin 4A}{4\cos\frac{1}{2}A\cos A\cos 2A}=\frac{2(2\sin A\cos A)\cos 2A}{4\cos\frac{1}{2}A\cos A\cos 2A}$$

$$=\frac{\sin A}{\cos\frac{1}{2}A}=2\sin\tfrac{1}{2}A:$$

$$\therefore c=2(a+b+c)\sin\tfrac{1}{2}A=\&\text{c.}$$

256. $\sin^2 C-\sin^2 A=\sin(C+A)\sin(C-A)$
$=\sin B\sin(C-A)=$ (by question) $\sin^2 B$;
$\therefore \sin(C-A)=\sin B$; hence $C-A=B$, or $=180^\circ-B$,
which last is impossible, since then $C+B=180^\circ+A>180^\circ$;
hence $C=A+B$, and $\therefore$ $C=90^\circ$.

257. $AB'^2=c'^2=a^2+b^2$, $AC'^2=b'^2=a^2+c^2$,
$BC'^2=a'^2=b^2+c^2$;

$$\therefore \cos BAC=\frac{b'^2+c'^2-a'^2}{2b'c'}=\frac{a^2}{\sqrt{\{(a^2+c^2)(a^2+b^2)\}}};$$

$$\therefore \sin^2 BAC=1-\frac{a^4}{(a^2+c^2)(a^2+b^2)}=\frac{a^2b^2+a^2c^2+b^2c^2}{(a^2+b^2)(a^2+c^2)};$$

and area $=\tfrac{1}{2}b'c'\sin BAC$.

258. If r angles of equal and similar figures, each of n sides, meet to make a solid angle, their sum $r\times\frac{n-2}{n}180^\circ$ must by (Euc. xi. 21.) be $<360^\circ$, whence $n<\frac{2r}{r-2}<2+\frac{4}{r-2}$: now this is greatest when $r-2$ is least, and neither r nor n can be <3; if $r=3$, $n<6$, and may, therefore, be 3, 4, or 5; if $r=4$, $n<4$, and must, therefore, be 3; if $r=5$, $n<3\tfrac{1}{3}$, and must *be 3; if* $r>5$, $n<3$, which is impossible; hence there are *only five* regular solids, namely, three in which each solid angle

is formed by *three* plane angles, where the faces may be *triangular*, *square*, or *pentagonal*, *one* in which each solid angle is formed by *four* plane angles, and *one* by *five*, in each of which the faces are *triangular*.

59. $c^2 = a^2 + b^2 - 2ab\cos C = (a+b)^2 - 4ab\cos^2\frac{1}{2}C$;
$\therefore 4ab = (m^2 - c^2)\sec^2\frac{1}{2}C = m^2\sin^2 2\phi$ suppose, which supposition is possible, since $2ab < a^2 + b^2$, and $\therefore 4ab < (a+b)^2$,
and we may put $4ab = (a+b)^2\sin^2 2\phi$:
then $m\sin 2\phi = \sqrt{(m^2 - c^2)}\sec\frac{1}{2}C$, and $2\sqrt{(ab)} = m\sin 2\phi$;
whence $a + b \pm 2\sqrt{(ab)} = m(1 \pm \sin 2\phi) = m(\cos\phi \pm \sin\phi)^2$,
or $\sqrt{a} \pm \sqrt{b} = \sqrt{m}(\cos\phi \pm \sin\phi)$;
$\therefore \sqrt{a} = \sqrt{m}\cos\phi$, $\sqrt{b} = \sqrt{m}\sin\phi$.

60. Let $\cos\theta = \frac{1}{2}(-1)^m = \pm\frac{1}{2}$, as m is even or odd;
(i) $\theta = 60°$, or $2n\pi \pm \frac{1}{3}\pi$; (ii) $\theta = 120°$, or $2n\pi \pm \frac{2}{3}\pi$:
both values of θ are included in $2n\pi \pm \{3 - (-1)^m\}\frac{1}{3}\pi$.

61. As in (253), $AC^2 + BC^2 = 2(AF^2 + CF^2)$,
or $b^2 + a^2 = 2(\frac{1}{4}c^2 + r^2)$;
$\therefore 4r^2 = 2(a^2 + b^2) - c^2 = 2(a^2 + b^2 + c^2) - 3c^2$;
and $4(p^2 + q^2 + r^2) = 6(a^2 + b^2 + c^2) - 3a^2 - 3b^2 - 3c^2$
$= 3(a^2 + b^2 + c^2)$;
and $c^2 = \frac{2}{3}(a^2 + b^2 + c^2) - \frac{4}{3}r^2 = \frac{8}{9}(p^2 + q^2 + r^2) - \frac{4}{3}r^2$
$= \frac{4}{9}\{2(p^2 + q^2) - r^2\}$.

62. $\sin A = 2\sin A\cos^2 A$;
$\therefore$ (i) $\sin A = 0$, or $A = n\pi$, (ii) $\cos A = \pm\frac{1}{\sqrt{2}}$,
or $A = 2n\pi + \frac{1}{4}\pi$, and $2n\pi + \frac{3}{4}\pi = (2n+1)\pi - \frac{1}{4}\pi$,
or, in one formula, $n\pi + (-1)^n\frac{1}{4}\pi$, where n may be even or odd. Hence chord $A = 2\sin\frac{1}{2}A$
$=$ (i) $2\sin\frac{1}{2}n\pi = 0$, or ± 2, as n is even or odd,
$=$ (ii) $2\sin\{\frac{1}{2}n\pi + (-1)^n\frac{1}{8}\pi\} = \pm 2\sin\frac{1}{8}\pi$, or $\pm 2\cos\frac{1}{8}\pi$,
according as n is even or odd,
$= \pm\sqrt{(2 - 2\cos\frac{1}{4}\pi)}$, or $\pm\sqrt{(2 + 2\cos\frac{1}{4}\pi)}$
$= \pm\sqrt{(2 - \sqrt{2})}$, or $\pm\sqrt{(2 + \sqrt{2})}$.

63. Here $a > b$, or else the expression is impossible: then
$\sqrt{a}\left\{\sqrt{\left(1 + \frac{b}{a}\right)} + \sqrt{\left(1 - \frac{b}{a}\right)}\right\}$
$= \sqrt{a}\{\sqrt{(1 + \cos\theta)} + \sqrt{(1 - \cos\theta)}\}$,
if $\cos\theta = \frac{b}{a} = \sqrt{a}\{\sqrt{2}\cos\frac{1}{2}\theta + \sqrt{2}\sin\frac{1}{2}\theta\}$
$= 2\sqrt{a}\{\cos 45°\cos\frac{1}{2}\theta + \sin 45°\sin\frac{1}{2}\theta\} =$ &c.

264. $AD^2 = AC^2 + CD^2 - 2AC.CD\cos\frac{1}{2}C$,
$BD^2 = BC^2 + CD^2 - 2BC.CD\cos\frac{1}{2}C$;
but (Euc. vi. 3.) $AD : BD :: AC : BC$,
or $AD^2.BC^2 = AC^2.BD^2$;
whence, using the above values of AD^2 and BD^2, we get
$CD^2(AC^2 \backsim BC^2) = 2(AC \backsim BC)AC.BC.CD\cos\frac{1}{2}C$,
or $CD =$ &c..

265. Let DE be the tower, E its foot in the horizontal plane:
then $DE = AE\tan\alpha = BE\tan\alpha = CE\tan\alpha$;
$\therefore AE = BE = CE$,
and E is the centre of the circle round ABC:
hence $AE = R =$ &c. (114),
and $DE = R\tan\alpha =$ &c., by (114).

266. Here $2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi) = \sin(\theta+\phi)$
$= 2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta+\phi)$;
$\therefore$ (i) $\sin\frac{1}{2}(\theta+\phi) = 0$, whence $\frac{1}{2}(\theta+\phi) = n\pi$,
or $\theta+\phi = n \times 2\pi = n \times 360°$;
or (ii) $\cos\frac{1}{2}(\theta-\phi) = \cos\frac{1}{2}(\theta+\phi)$,
whence $\frac{1}{2}(\theta+\phi) = 2n\pi \pm \frac{1}{2}(\theta-\phi)$,
or $\frac{1}{2}\{(\theta+\phi) \mp (\theta-\phi)\}$, that is, θ or ϕ, $= 2n\pi = n \times 360°$.

267. Suppose $a > b$; then $CD = b\dfrac{\sin A}{\sin\alpha}$, $CE = a\dfrac{\sin B}{\sin\beta}$,

$\angle DCE = \alpha - \beta$;

$$\therefore \text{area} = \tfrac{1}{2}CD.CE\sin DCE = \tfrac{1}{2}ab\frac{\sin A\sin B}{\sin\alpha\sin\beta}\sin(\alpha-\beta)$$

$$= \tfrac{1}{2}ab\frac{a}{c}.\frac{b}{c}\;\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} = \text{\&c.}$$

268. As in (116), $2\cos^2\frac{1}{2}A = 1 + \cos A = \dfrac{(s-b)(s-c)}{2(ad+bc)}$,

$2\sin^2\frac{1}{2}A = \dfrac{(s-a)(s-d)}{2(ad+bc)}$; $\therefore \tan\frac{1}{2}A =$ &c.

If CD, or c, vanishes, so that C coincides with D,
then, using a', b', c', for the sides of the resulting triangle, we have $AB = a = c'$, $BC = b = a'$, $CD = c = 0$,
DA or $CA = d = b'$, and $s = a' + b' + c' = s'$;

$\therefore \tan\frac{1}{2}A = \sqrt{\dfrac{(s'-c')(s'-b')}{(s'-a')s'}}$, as it should be.

269. $u = \log_e \dfrac{1}{2\sin^2 \frac{1}{2}\theta} = \log_e \dfrac{1}{1-\cos\theta}$

$= -\log_e (1-\cos\theta) =$ &c., by (150).

270. A's share being to the whole as $m : m+n+p$, take in AB a length $AD : AB :: \surd m : \surd(m+n+p)$, and draw DE parallel to BC;

then area ADE : area $ABC :: AD^2 : AB^2 :: m : m+n+p$,
that is, area ADE will be A's share.
So if $AF : AB :: \surd(m+n) : \surd(m+n+p)$,
and FG be drawn parallel to BC,
then area AFG will be A's share + B's share;
that is, area $DEFG$ will be B's,
and, therefore, area $BCGF$ will be C's share.

271. (i) $u = \dfrac{1}{\sin 2A + 1} = \dfrac{1}{(\cos A + \sin A)^2} = \dfrac{1}{\cos^2 A(1+\tan A)^2}$

$= \dfrac{\sec^2 A}{(1+\tan A)^2}$:

(ii) $u = \dfrac{1}{\sin A} = \dfrac{2\cos^2 \frac{1}{2}A - \cos A}{\sin A} = \cot \frac{1}{2}A - \cot A.$

72. $\Sigma = \left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \text{\&c.}\right) - \left(\dfrac{1}{3^{\frac{1}{2}}} - \dfrac{1}{3.3^{\frac{3}{2}}} + \dfrac{1}{5.3^{\frac{5}{2}}} - \text{\&c.}\right)$

$=$ by [8] $\dfrac{\pi}{4} - \dfrac{\pi}{6} = \dfrac{\pi}{12}.$

73. $\sin A = \sin\{n\pi + (-1)^n A\}$ $\therefore$ $\sin mA$ will give the general value for $\sin\{mn\pi + (-1)^n mA\}$, that is, (expanding by the formula for $\sin(A+B)$, and observing that $\sin mn\pi = 0$), it will give the general value for $\cos mn\pi \sin(-1)^n mA$,
which $= (-1)^n \sin mA$, if m be *even*,
and this has the two values $\pm \sin mA$,
or $= (-1)^n \sin(-1)^n mA = (-1)^{2n} \sin mA = + \sin mA$, if m be odd.
Again, $\cos A = \cos(2n\pi \pm A)$;
$\therefore$ $\cos mA$ will give $\cos(2mn\pi \pm mA) = + \cos mA$, whether m be even or odd.

74. $2 + \dfrac{1}{\sec^2\theta - 1} = 3\sec^4\theta - \sec^2\theta + 1$, whence $3\sec^6\theta = 4\sec^4\theta$,

which gives $\sec^2\theta = 0$ (*impossible*), or $\cos\theta = \pm \frac{1}{2}\surd 3$,
whence $\theta = 30°$ or $150°$,
or, generally, $2n\pi + \frac{1}{6}\pi$, or $2n\pi \pm \frac{5}{6}\pi$.

5. [illegible] $-\theta$, where θ is very small;

[illegible] $\Big\}\cot\theta = \dfrac{\theta}{\tan\theta}$ nearly,

[illegible] when $\theta = 0$.

6. [illegible] $EC = 48 = EF + BC - BF$

$= 30\sin 75^\circ + 30\sin 60^\circ - x\sin 60^\circ,$

[illegible] $BD = x$;

$$\therefore\ x = 30\frac{\sin 75^\circ}{\sin 60^\circ} + 30 - \frac{48}{\sin 60^\circ}$$

$$= 30\frac{\sqrt{3}-1}{\sqrt{6}} + 30 - \frac{96}{\sqrt{3}}$$

$$= 15\sqrt{2} - 5\sqrt{6} + 30 - 32\sqrt{3}$$

[illegible] $+ 30 - 32(1.732) = 8$ *ft.*, nearly.

7. [illegible] $= (4x-3)\log 2 + x\log 5$;

whence [illegible] $= 6\log 5 - 3\log 2$,

[illegible] $= 1.062$.

8. [illegible] be the centres of squares on BC, AC:

then [illegible] $C\,.\,B'C\cos 150^\circ$

[illegible] $-\cos 150^\circ) = a^2(1 + \frac{1}{2}\sqrt{3})$;

and since the triangles, being both equilateral, will be similar,

we have their areas as $a^2(1+\frac{1}{2}\sqrt{3}) : a^2$, or as $\sqrt{3}+2 : 2$.

9. [illegible] $= a^2 + d^2 - ad\dfrac{a^2+d^2-b^2-c^2}{ad+bc}$

[illegible]

and radius of circumscribing circle $= \dfrac{BD}{2\sin A} =$ &c., by (116).

10. [illegible] $-1 = 0$,

whence [illegible] $(\cos^2 a - \frac{1}{2})$:

[illegible] $= \cos a\sqrt{2} = \cos a \sec 45^\circ$;

[illegible] $) = \cos 45^\circ\,\dfrac{\sin 2\phi \pm 1}{\cos 2\phi}$

[illegible] $= \pm\cos 45^\circ\,\dfrac{\cos\phi \pm \sin\phi}{\cos\phi \mp \sin\phi}$

$$=\cos 45^\circ \frac{1+\tan\phi}{1-\tan\phi}, \text{ or } \cos 45^\circ \frac{\tan\phi-1}{1+\tan\phi}=\&\text{c}.$$

81. By (115) $r_1=\dfrac{S}{s-a}$;

$$\therefore r_1 \cot\tfrac{1}{2}A=\frac{S}{s-a}\sqrt{\frac{s(s-a)}{(s-b)(s-c)}},$$

where $S=\sqrt{\{s(s-a)(s-b)(s-c)\}}$ by (109);
$\therefore r_1\cot\frac{1}{2}A=s=r_2\cot\frac{1}{2}B=r_3\cot\frac{1}{2}C$:
also $\surd(r_1r_2+r_1r_3+r_2r_3)$
$=\sqrt{\{s(s-c)+s(s-b)+s(s-a)\}}$
$=\sqrt{\{s(3s-a-b-c)\}}=s.$

82. By (118) the ratio is that of $\cos^2 18^\circ : 1=\frac{1}{8}(5+\sqrt{5})$;

$$\text{now } \sqrt{5}=2+(\sqrt{5}-2)=2+\frac{1}{\sqrt{5}+2}$$

$$=2+\frac{1}{4+}\;\frac{1}{4+\&\text{c.}} \text{ by Colenso's Alg. Part II. (154)};$$

$$\therefore \frac{1}{8}(5+\sqrt{5})=\frac{1}{8}\left(7+\frac{1}{4+}\;\frac{1}{4+}\;\ldots\right)=\frac{7}{8} \text{ or } \frac{29}{32} \text{ or } \frac{123}{136}, \&\text{c.}$$

83. $q-p=p(\cos^4\theta-\sin^4\theta)-q(\cos^4\phi-\sin^4\phi)$
$=p(\cos^2\theta-\sin^2\theta)-q(\cos^2\phi-\sin^2\phi)$:
$\therefore p\cos^2\theta=q\cos^2\phi$:

hence from (2) $\cos^4\theta\left(p-\dfrac{p^2}{q}\right)=q$, or $\cos^4\theta=\&$c.

84. A, B, the two points, CD the tower,
$\angle CAD=\alpha$, $\angle CBD=\beta$, $\angle ACB=\theta$;
put $CD=h$: then $AC=h\operatorname{cosec}\alpha$, $BC=h\operatorname{cosec}\beta$,
$a^2=h^2\operatorname{cosec}^2\alpha+h^2\operatorname{cosec}^2\beta-2h^2\operatorname{cosec}\alpha\operatorname{cosec}\beta\cos\theta$;

$$\therefore h=\frac{a}{\sqrt{(\operatorname{cosec}^2\alpha+\operatorname{cosec}^2\beta-2\operatorname{cosec}\alpha\operatorname{cosec}\beta\cos\theta)}}$$

$$=\frac{a\sin\alpha\sin\beta}{\sqrt{(\sin^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta\cos\theta)}}$$

$$=\frac{a\sin\alpha\sin\beta}{\sqrt{\{(\sin\alpha-\sin\beta)^2+4\sin\alpha\sin\beta\sin^2\frac{1}{2}\theta\}}}$$

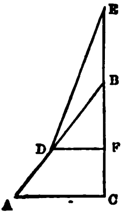
$$=\frac{a\sin\alpha\sin\beta}{2\sqrt{\{\sin^2\frac{1}{2}(\alpha-\beta)\cos^2\frac{1}{2}(\alpha+\beta)+\sin\alpha\sin\beta\sin^2\frac{1}{2}\theta\}}}$$

275. Put $x=\frac{1}{2}\pi-\theta$, where θ is very small;

then $\left\{\frac{\pi}{2}-\theta-\frac{2}{\pi}\left(\frac{\pi^2}{4}-\pi\theta-\theta^2\right.\right\}\cot\theta=\frac{\theta}{\tan\theta}$ nearly,

(omitting θ^2,) $=1$, by (91), when $\theta=0$.

276.



$EC=48=EF+BC-BF$

$=30\sin 75^\circ+30\sin 60^\circ-x\sin 60^\circ$,

if $BD=x$;

$\therefore\ x=30\frac{\sin 75^\circ}{\sin 60^\circ}+30-\frac{48}{\sin 60^\circ}$

$=30\frac{\sqrt{3}+1}{\sqrt{6}}+30-\frac{96}{\sqrt{3}}$

$=15\sqrt{2}+5\sqrt{6}+30-32\sqrt{3}$

$=15(1.414)+5(2.449)+30-32(1.732)=8$ *ft.*, nearly.

277. $x(3\log 2)+(2-x)3\log 5=(4x+3)\log 2+x\log 5$;
whence $x(\log 2+4\log 5)=6\log 5-3\log 2$,

or $x=\frac{6-9\log 2}{4-3\log 2}=\frac{3.29073}{3.09691}=1.062$.

278. Let A', B', be the centres of squares on BC, AC:
then $A'B'^2=A'C^2+B'C^2-2A'C.B'C\cos 150^\circ$
$=2A'C^2(1-\cos 150^\circ)=a^2(1+\frac{1}{2}\sqrt{3})$;
and since the triangles, being both equilateral, will be similar, we have their areas as $a^2(1+\frac{1}{2}\sqrt{3}):a^2$, or as $\sqrt{3}+2:2$.

279. $BD^2=a^2+d^2-2ad\cos A=a^2+d^2-ad\frac{a^2+d^2-b^2-c^2}{ad+bc}$

$=\frac{(ac+bd)(ab+cd)}{ad+bc}$;

and radius of circumscribing circle $=\frac{BD}{2\sin A}=$&c., by (116).

280. Here $2\cos^2\theta-4\cos a\cos\theta-1=0$,
whence $\cos\theta=\cos a\pm\sqrt{(\cos^2 a+\frac{1}{2})}$:
put $\cos^2 a=\frac{1}{2}\tan^2 2\phi$, or $\tan 2\phi=\cos a\sqrt{2}=\cos a\sec 45^\circ$;

then $\cos\theta=\frac{1}{\sqrt{2}}(\tan 2\phi\pm\sec 2\phi)=\cos 45^\circ\ \frac{\sin 2\phi\pm 1}{\cos 2\phi}$

$=\pm\cos 45^\circ\frac{(\cos\phi\pm\sin\phi)^2}{\cos^2\phi-\sin^2\phi}=\pm\cos 45^\circ\frac{\cos\phi\pm\sin\phi}{\cos\phi\mp\sin\phi}$

$$=\cos 45^\circ \frac{1+\tan\phi}{1-\tan\phi}, \text{ or } \cos 45^\circ \frac{\tan\phi-1}{1+\tan\phi} = \&c.$$

281. By (115) $r_1 = \frac{S}{s-a}$;

$$\therefore r_1 \cot \tfrac{1}{2}A = \frac{S}{s-a}\sqrt{\frac{s(s-a)}{(s-b)(s-c)}},$$

where $S=\sqrt{\{s(s-a)(s-b)(s-c)\}}$ by (109);
$\therefore r_1 \cot\frac{1}{2}A = s = r_2 \cot\frac{1}{2}B = r_3 \cot\frac{1}{2}C$:
also $\sqrt{(r_1 r_2 + r_1 r_3 + r_2 r_3)}$
$=\sqrt{\{s(s-c)+s(s-b)+s(s-a)\}}$
$=\sqrt{\{s(3s-a-b-c)\}}=s.$

282. By (118) the ratio is that of $\cos^2 18^\circ : 1 = \frac{1}{8}(5+\sqrt{5})$;

$$\text{now } \sqrt{5} = 2+(\sqrt{5}-2) = 2+\frac{1}{\sqrt{5}+2}$$

$$=2+\frac{1}{4+}\,\frac{1}{4+\&c.} \text{ by Colenso's Alg. Part II. (154)};$$

$$\therefore \frac{1}{8}(5+\sqrt{5}) = \frac{1}{8}\left(7+\frac{1}{4+}\,\frac{1}{4+}\ \ldots\right) = \frac{7}{8} \text{ or } \frac{29}{32} \text{ or } \frac{123}{136}, \&c.$$

283. $q-p = p(\cos^4\theta - \sin^4\theta) - q(\cos^4\phi - \sin^4\phi)$
$=p(\cos^2\theta-\sin^2\theta) - q(\cos^2\phi - \sin^2\phi)$:
$\therefore p\cos^2\theta = q\cos^2\phi$:

hence from (2) $\cos^4\theta\left(p-\frac{p^2}{q}\right)=q$, or $\cos^4\theta = \&c.$

284. A, B, the two points, CD the tower,
$\angle CAD = \alpha$, $\angle CBD = \beta$, $\angle ACB = \theta$;
put $CD = h$: then $AC = h\operatorname{cosec}\alpha$, $BC = h\operatorname{cosec}\beta$,
$a^2 = h^2\operatorname{cosec}^2\alpha + h^2\operatorname{cosec}^2\beta - 2h^2\operatorname{cosec}\alpha\operatorname{cosec}\beta\cos\theta$;

$$\therefore h = \frac{a}{\sqrt{(\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta - 2\operatorname{cosec}\alpha\operatorname{cosec}\beta\cos\theta)}}$$

$$=\frac{a\sin\alpha\sin\beta}{\sqrt{(\sin^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta\cos\theta)}}$$

$$=\frac{a\sin\alpha\sin\beta}{\sqrt{\{(\sin\alpha-\sin\beta)^2+4\sin\alpha\sin\beta\sin^2\frac{1}{2}\theta\}}}$$

$$=\frac{a\sin\alpha\sin\beta}{2\sqrt{\{\sin^2\frac{1}{2}(\alpha-\beta)\cos^2\frac{1}{2}(\alpha+\beta)+\sin\alpha\sin\beta\sin^2\frac{1}{2}\theta\}}}$$

put $\sin a \sin \beta \sin^2 \frac{1}{2}\theta = \sin^2 \frac{1}{2}(a-\beta)\cos^2\frac{1}{2}(a+\beta)\tan^2\phi$;

then $h = \dfrac{a \sin a \sin \beta \cos \phi}{2 \sin \frac{1}{2}(a-\beta)\cos\frac{1}{2}(a+\beta)}$.

285. Let the inscribed circle touch the sides AB, BC, &c., in E, F, G, H:

then $AE = AH$, $BE = BF$, $CG = CF$, $DG = DH$;

$\therefore (AE+BE)+(CG+DG) = (AH+DH)+(BF+CF)$,

or $a+c=b+d$.

Hence $s = a+c = b+d$,

and area $= \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}} = \sqrt{(cdab)}$

$= \sqrt{(abcd)} =$ also, (by breaking it up into four triangles, whose common vertex is the centre of inscribed circle, radius r,) $\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr + \frac{1}{2}dr = sr$;

$\therefore sr =$ &c.

286. (i) $\cos A = \cos(2n\pi \pm A)$;

$\therefore$ the value of $\sin\frac{1}{2}A$ in terms of $\cos A$ will include those of

$\sin\frac{1}{2}(2n\pi \pm A) = \sin(n\pi \pm \frac{1}{2}A) = \pm \cos n\pi \sin \frac{1}{2}A$

$= \pm \sin \frac{1}{2}A$:

(ii) $\sin A = \sin\{n\pi + (-1)^n A\}$;

$\therefore$ the value of $\sin\frac{1}{2}A$ in terms of $\sin A$ will include those of $\sin\{\frac{1}{2}n\pi + (-1)^n \frac{1}{2}A\}$, which (if n be even, and $=2m$) gives $\cos m\pi \sin\frac{1}{2}A = \pm \sin\frac{1}{2}A$, according as m is even or odd:

or (if n be odd, and $=2m+1$) it gives

$\sin(m\pi + \frac{1}{2}\pi - \frac{1}{2}A) = \cos m\pi \sin(\frac{1}{2}\pi - \frac{1}{2}A) = \pm \cos\frac{1}{2}A$.

287. By (153), $\log_{10}(n+1) = \log n + 2M\left\{\dfrac{1}{2n+1} + \text{\&c.}\right\}$,

or $\log_{10} 255 = \log_{10} 256 - 2M\{\frac{1}{511} + \text{\&c.}\}$

$= 8\log_{10} 2 - 2M \times .0019569$

$= 2.4082400 - .0016996 = 2.4065404$.

288. $\cos(a+\theta) = \sin(a-\theta) = \cos(\frac{1}{2}\pi - a + \theta)$;

$\therefore a+\theta = 2n\pi \pm (\frac{1}{2}\pi - a + \theta)$:

this is impossible, with the upper sign, unless $a = (4n+1)\frac{1}{4}\pi$, when it is true for all values of θ;

with the lower sign, $\theta = (4n-1)\frac{1}{4}\pi$.

289. $\tan\theta \sin\psi = \tan\theta \sin(\phi+\psi-\phi)$

$= \tan\theta\{\sin(\phi+\psi)\cos\phi - \cos(\phi+\psi)\sin\phi\}$

$=$ (by question) $\sin\phi \sin(\phi+\psi)$;

$\therefore \sin\phi\{\sin(\phi+\psi)\cos\theta + \sin\theta\cos(\phi+\psi)\}$,

or $\sin\phi \sin(\theta+\phi+\psi) = \sin\theta \sin(\phi+\psi)\cos\phi$; $\therefore \tan\phi =$ &c.

296. If a be the side of an equilateral triangle, and R, r, the radii of its circumscribed and inscribed circles,

then $R=\frac{a}{2\sin A}=\frac{a}{2\sin 60^\circ}=\frac{a}{\sqrt{3}}$, and $r=\frac{a}{2}\tan 30^\circ=\frac{a}{2\sqrt{3}}$:

hence, using r_1, r_2, &c. for the radii of the inner circles,

we have $r_1=\frac{a}{2\sqrt{3}}=\frac{1}{2}R$, $r_2=\frac{1}{2}r_1$, &c.

$\therefore\ 2\pi(r_1+r_2+\&c.)=\pi R(1+\frac{1}{2}+\&c.)=2\pi R$,

and $\pi(r_1{}^2+r_2{}^2+\&c.)=\frac{1}{4}\pi R^2(1+\frac{1}{4}+\&c.)=\frac{1}{3}\pi R^2$.

291. $\sin\theta-\cos\theta=4\sin\theta-4\sin^3\theta$,

or $\cos\theta=4\sin^3\theta-3\sin\theta=-\sin 3\theta=\cos(\frac{1}{2}\pi+3\theta)$:

$\therefore\ \frac{1}{2}\pi+3\theta=2n\pi\pm\theta$, &c.

292. Let the angles be 2θ, 3θ, 4θ; then $9\theta=180^\circ$, and the angles are 40°, 60°, 80° ;

hence $\sin C=\sin 2A$, or $\frac{\sin C}{\sin A}=2\cos A$,

that is, $\frac{c}{a}=\frac{b^2+c^2-a^2}{bc}$,

whence $c^2(a-b)=a(a^2-b^2)$, or $c^2=a(a+b)$: (1)

Also $\sin\frac{1}{2}B=\frac{1}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}$, or $\sqrt{\frac{s-a}{c}}=\frac{1}{2}\sqrt{\frac{a}{s-c}}$,

and $\therefore\ \cos\frac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}=\frac{1}{2}\sqrt{\frac{as}{b(s-c)}}$

$$=\frac{1}{2}\sqrt{\frac{a(a+b+c)}{b(a+b-c)}}=\frac{a+c}{2b};$$

for $\frac{a+b}{c}=\frac{c}{a}$ from (1), and $\therefore\ \frac{a+b+c}{a+a-c}=\frac{c+a}{c-a}$,

and $\frac{a}{b}\cdot\frac{a+b+c}{a+b-c}$, $=$ from (1), $\frac{c^2-a^2}{b^2}\cdot\frac{c+a}{c-a}=\frac{(c+a)^2}{b^2}$.

93. Join the centres of the circles, A, B, C, and let D, E, F, be the points of contact, opposite to the angles A, B, C, respectively: then, since the two tangents, drawn to each circle from the same point, are equal, it follows that the three tangents are equal ($=p$, suppose); that is, the tangents at D and E will cut off each the *same* length from the tangent at F, or the *three tangents* will meet in the same point P. Let a', b', c'.

be the sides of the triangle ABC; then (since the tangents are perp. to the sides of the triangle)

we have area $ABC = \frac{1}{2}(a'+b'+c')p = (a+b+c)p$,

(since $a' = b+c$, $b' = a+c$, $c' = a+b$,)

$$= \sqrt{\{s'(s'-a')(s'-b')(s'-c')\}}$$

$$= \sqrt{\{(a+b+c)abc\}}; \text{ hence } p = \&\text{c.}$$

The point of intersection of the tangent is, in fact, the centre of the circumscribing circle of the triangle ABC, since the angles A, B, C, will be bisected by the lines AP, BP, CP.

294. O the centre of square, O_1 the centre of first circle inscribed in corner A, O_1B its radius perp. to side of square, OC the radius of circle, centre O, drawn parallel to O_1B: then, joining OO_1A, we have $OC = OO_1 \sin 45° + O_1B$,

$$\text{or } r = (r+r_1)\frac{1}{\sqrt{2}} + r_1, \text{and } r_1 = \frac{\sqrt{2}-1}{\sqrt{2}+1}r; \text{ so } r_2 = \frac{\sqrt{2}-1}{\sqrt{2}+1}r_1, \&\text{c.};$$

and sum of perimeters: that of circle, centre O

$$:: 4\left\{\frac{\sqrt{2}-1}{\sqrt{2}+1} + \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 + \&\text{c.}\right\} : 1$$

$$= 4\frac{\sqrt{2}-1}{\sqrt{2}+1} \cdot \frac{1}{1-\frac{\sqrt{2}-1}{\sqrt{2}+1}} = 2(\sqrt{2}-1) : 1;$$

and sum of areas : that of circle, centre O

$$:: 4(r_1{}^2 + r_2{}^2 + \&\text{c.}) : r^2$$

$$= 4\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 \cdot \frac{1}{1-\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2} = 4\frac{(\sqrt{2}-1)^2}{4\sqrt{2}} = 3-2\sqrt{2} : \sqrt{2}.$$

295. $OA^2 + OB^2 + OC^2 = r^2(\text{cosec}^2 \frac{1}{2}A + \text{cosec}^2 \frac{1}{2}B + \text{cosec}^2 \frac{1}{2}C)$

$$= \frac{S^2}{s^2}\left\{\frac{bc}{(s-b)(s-c)} + \frac{ac}{(s-a)(s-c)} + \frac{ab}{(s-a)(s-b)}\right\} \text{ by (115)}$$

and (107)

$$= \frac{bc(s-a) + ac(s-b) + ab(s-c)}{s} = bc + ac + ab - \frac{3abc}{s}.$$

296. (i) $a > b$; $u = \dfrac{1}{a(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) + b(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x)}$

$$=\frac{\sec^2\frac{1}{2}x}{(a+b)+(a-b)\tan^2\frac{1}{2}x}=\frac{\sec^2\frac{1}{2}x\cos^2\phi}{a+b},$$

if $(a-b)\tan^2\frac{1}{2}x=(a+b)\tan^2\phi$,

or $\tan^2\phi=\frac{a-b}{a+b}\tan^2\frac{1}{2}x=\frac{1-\cos\theta}{1+\cos\theta}\tan^2\frac{1}{2}x\left(\text{if }\cos\theta=\frac{b}{a}\right)$

$=\tan^2\frac{1}{2}\theta\tan^2\frac{1}{2}x$;

hence $u=\frac{\sec^2\frac{1}{2}x\cos^2\phi}{a(1+\cos\theta)}=\frac{1}{2a}\sec^2\frac{1}{2}x\cos^2\phi\sec^2\frac{1}{2}\theta$:

(ii) $a<b$; $u=\frac{1}{b(\cos\theta+\cos x)}\left(\text{if }\cos\theta=\frac{a}{b}\right)$

$$=\frac{1}{2b\cos\frac{1}{2}(\theta+x)\cos\frac{1}{2}(\theta-x)}.$$

97. By [6] $u=\frac{2}{3}[8\{\frac{1}{4}\theta-\frac{1}{6}(\frac{1}{4}\theta)^3\}-\{\frac{1}{2}\theta-\frac{1}{6}(\frac{1}{2}\theta)^3\}]+$ terms in θ^5 &c.
$=\frac{2}{3}(2\theta-\frac{1}{48}\theta^3-\frac{1}{2}\theta+\frac{1}{48}\theta^3)+$ terms in θ^5 &c.
$=\theta+$ terms in θ^5 &c.
$=\theta$ very nearly, when θ is small.

98. Let $CD=c$, radius $=1$;
then $OD=OA\tan OAD$,
or $c+1=\tan OAB=\tan 60°=\sqrt{3}$;
$\therefore\ c=\sqrt{3}-1=\text{chord}\,\theta=2\sin\frac{1}{2}\theta=\sqrt{(2-2\cos\theta)}$;
$\therefore\ 2-2\cos\theta=4-2\sqrt{3}$, or $\cos\theta=\sqrt{3}-1=\text{chord}\ \theta$.

99. Let $\theta=\tan^{-1}\frac{4(x-x^3)}{1+x^2}$, $\phi=\text{chd}^{-1}\frac{2}{\sqrt{(1+x^2)}}$:

then $\theta=2\pi-2\phi$, and $\tan\theta=-\tan 2\phi$,

or $\frac{4(x-x^3)}{1+x^2}=\frac{4x(1-x^2)}{x^4-6x^2+1}$,

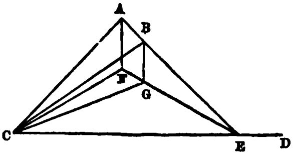
for $\text{chd}\,\phi=\frac{2}{\sqrt{(1+x^2)}}=2\sin\frac{1}{2}\phi$;

$\therefore\ \sin\frac{1}{2}\phi=\frac{1}{\sqrt{(1+x^2)}}$, $\cos\frac{1}{2}\phi=\frac{x}{\sqrt{(1+x^2)}}$, $\tan\frac{1}{2}\phi=\frac{1}{x}$,

$\tan\phi=\frac{2x}{x^2-1}$, $\tan 2\phi=$&c.:

hence $x=0$, or $1-x^2=0$, or $x^4-6x^2+1=1+x^2$;
$\therefore\ x=0$, or ± 1, or $\pm\sqrt{7}$.

300.



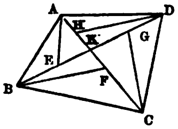
$\angle ACF = \alpha$, $\angle BCG = \beta$, $\angle ACB = \gamma$, $\angle ACE = \delta$, $\angle AEC = \epsilon$, $CE = a$: $AF = h$, $BG = k$; in ΔACE, we have CE and $\angle\angle CE$;

hence $AC = a\dfrac{\sin\epsilon}{\sin(\delta+\epsilon)}$, and $h = AC\sin\alpha = a\dfrac{\sin\alpha\sin\epsilon}{\sin(\delta+\epsilon)}$:

so in ΔBCE, we have CE, $\angle E$, and $\angle BCE = \delta - \gamma$;

$\therefore BC = a\dfrac{\sin\epsilon}{\sin(\delta-\gamma+\epsilon)}$, and $k = BC\sin\beta$.

301.



Let $\angle AEK = \angle BFK = \angle CGK = \angle DHK = \theta$, K being the point of intersection of the diagonals:

then $AE = AB\dfrac{\sin ABD}{\sin\theta}$,

$BF = BC\dfrac{\sin BCA}{\sin\theta} = BC\dfrac{\sin BDA}{\sin\theta}$, $CG = CD\dfrac{\sin CDB}{\sin\theta}$,

$DH = AD\dfrac{\sin DAC}{\sin\theta} = AD\dfrac{\sin DBC}{\sin\theta}$;

$$\therefore \frac{AE.CG}{BF.DH} = \frac{AB.CD}{BC.AD}\times\frac{\sin ABD}{\sin BDA}\times\frac{\sin CDB}{\sin DBC}$$

$$= \frac{ac}{bd}\times\frac{d}{a}\times\frac{b}{c} = 1.$$

302. $\sin(\theta+\phi+\psi) = 1 = \sin(\theta+\phi)\cos\psi + \cos(\theta+\phi)\sin\psi$,
whence, expanding, we get
$\sin\theta\cos\phi\cos\psi + \sin\phi\cos\theta\cos\psi + \sin\psi\cos\theta\cos\phi$
$= 1 + \sin\theta\sin\phi\sin\psi$,
or $\tan\theta + \tan\phi + \tan\psi = \sec\theta\sec\phi\sec\psi + \tan\theta\tan\phi\tan\psi$:
again $\cos(\theta+\phi+\psi) = 0$,
whence, as before, $\cot\theta + \cot\phi + \cot\psi = \cot\theta\cot\phi\cot\psi$:
multiply now together the corresponding sides of these two results.

303. $\sin^2 2\theta - \sin^2\theta = 4\sin^2\theta\cos^2\theta - \sin^2\theta = 3\sin^2\theta - 4\sin^4\theta = \frac{1}{4}$;

whence $\sin^2\theta = \dfrac{6 \pm 2\sqrt{5}}{16} = \sin^2 18^\circ$ or $\sin^2 54^\circ$;

$\therefore \sin\theta = \pm \sin 18^\circ$ or $\pm \sin 54^\circ$,

and $\theta = (n\pi \pm \frac{1}{10}\pi)$ or $(n\pi \pm \frac{3}{10}\pi) = (n \pm \frac{1}{5} \pm \frac{1}{10})\pi$.

304. $u = 5^\circ.729577$,

and $\therefore$ its circular measure $(\alpha) = \dfrac{5.729577}{57.29577} = \dfrac{1}{10}$;

hence $\sin\alpha = .1 - \frac{1}{6}(.001) + \frac{1}{120}(.00001) - \&c.$

$= .1 - .0001666 + .00000008 = .0998334$.

305. $\cot A = \cot(n\pi + A)$;

hence the value of $\sin\frac{1}{3}A$, in terms of $\cot A$, will give the value of $\sin\frac{1}{3}(n\pi + A)$;

now n may be of the form $3m$ or $3m \pm 1$;

(i) $n = 3m$ gives $\sin(m\pi + \frac{1}{3}A) = \pm \sin\frac{1}{3}A$, as m is even or odd;

(ii) $n = 3m + 1$ gives $\sin\{m\pi + \frac{1}{3}(\pi + A)\} = \pm \sin\frac{1}{3}(\pi + A)$;

(iii) $n = 3m - 1$ gives $\sin\{m\pi - \frac{1}{3}(\pi - A)\} = \mp \sin\frac{1}{3}(\pi - A)$.

06. Let $\sin\alpha = 2x$, $\sin\beta = x\sqrt{3}$;

then, by question, $\alpha - \beta = \sin^{-1}x$,

or $x = \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= 2x\sqrt{(1 - 3x^2)} - x\sqrt{3}\sqrt{(1 - 4x^2)}$,

or $1 + \sqrt{(3 - 12x^2)} = \sqrt{(4 - 12x^2)}$;

squaring, we shall find $2\sqrt{(3 - 12x^2)} = 0$, or $x = \pm\frac{1}{2}$.

07. $\tan C = -\tan(A + B) = \dfrac{\tan A + \tan B}{\tan A \tan B - 1}$;

$\therefore \dfrac{\tan A}{\tan C} + \dfrac{\tan B}{\tan C} = \tan A \tan B - 1$,

and $u = (\tan A \tan B + \tan A \tan C + \tan B \tan C) - 3$

$= (1 + \sec A \sec B \sec C) - 3 = \&c.$;

for $\cos(A + B + C) = -1$

$= \cos(A + B)\cos C - \sin(A + B)\sin C$

$= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \sin C \cos B$

$- \sin B \sin C \cos A$;

$\therefore$ dividing every term by $\cos A \cos B \cos C$, we get

$\tan A \tan B + \tan A \tan C + \tan B \tan C = 1 + \sec A \sec B \sec C$.

08. C the centre, draw PM perp. to AB; then

$$2\sin\theta = (\sin\theta + \cos\theta) + (\sin\theta - \cos\theta) = \frac{BP + AP}{AB} + \frac{BP - AP}{AB}$$

$$= \sqrt{\frac{(BP + AP)^2}{AB^2}} + \sqrt{\frac{(BP - AP)^2}{AB^2}}$$

$$= \sqrt{\left(1 + \frac{BP.AP}{2CP^2}\right)} + \sqrt{\left(1 - \frac{BP.AP}{2CP^2}\right)};$$

but $BP.AP=\surd(AB.BM)\times\surd(AB.AM)$
$=AB.PM=2CP.PM$;

$\therefore\ 2\sin\theta=\surd\left(1+\frac{PM}{PC}\right)+\surd\left(1-\frac{PM}{PC}\right)$
$=\surd(1+\sin2\theta)+\surd(1-\sin2\theta)$.

See also the first part of Ex. 359.

309. $\frac{\cos(\beta-\theta)}{\sin\theta}=\frac{1+x}{2}\sin\beta=\cos\beta\cot\theta+\sin\beta$;

$\therefore\ \tan\theta=\frac{2}{x-1}\cot\beta$;

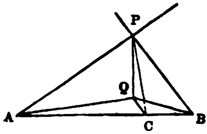
also, by question, $\tan(\theta-\alpha)=\frac{2}{x+1}\cot\beta$;

$\therefore\ \tan\alpha=\tan\{\theta-(\theta-\alpha)\}$
$=\frac{\tan\theta-\tan(\theta-\alpha)}{1+\tan\theta\tan(\theta-\alpha)}=\frac{4\cot\beta}{(x^2-1)+4\cot^2\beta}$;

or $x^2=1+4\cot\beta\cot\alpha-4\cot^2\beta$
$=1+2\cot\beta(\cot\frac{1}{2}\alpha-\tan\frac{1}{2}\alpha)-4\cot^2\beta$
$=(\cot\frac{1}{2}\alpha-2\cot\beta)(\tan\frac{1}{2}\alpha+2\cot\beta)$.

310. $u=\frac{2\sin2\theta}{2\theta}\cdot\frac{2\sin6\theta\cos6\theta}{\sin3\theta}=8\frac{\sin2\theta}{2\theta}\cos3\theta\cos6\theta=0$;

hence (i) $\sin2\theta=0$, (except when $2\theta=0$,)
or (ii) $\cos3\theta=0$,
or (iii) $\cos6\theta=0$;
(i) gives $2\theta=n\pi$, (except when $n=0$,)
(ii) gives $3\theta=(2n+1)\frac{1}{2}\pi$,
(iii) gives $6\theta=(2n+1)\frac{1}{2}\pi$;
$\therefore\ \theta=\frac{1}{2}n\pi$, (except when $n=0$,) or $=(2n+1)(1\pm\frac{1}{3})\frac{1}{8}\pi$.

311. 

AP, BP, the given lines at right angles to each other, cut by AB, a horizontal line upon the face of the stratum; PQ a perp. on the hor. plane through AB, QC perp. on AB:

given $\angle PAQ=\alpha$, $\angle PBQ=\beta$, to find $\angle PCQ=\theta$:
put $PQ=p$; then $AP=p\operatorname{cosec}\alpha$, $BP=p\operatorname{cosec}\beta$,

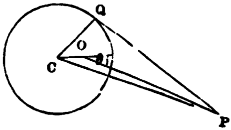
and $\tan PBA=\frac{AP}{BP}=\frac{\sin\beta}{\sin\alpha}=\tan\phi$, suppose:

now $PC = PB \sin\phi = p \sin\phi \operatorname{cosec}\beta$,

and $\sin\theta = \dfrac{PQ}{PC} = \sin\beta \operatorname{cosec}\phi$;

$$\therefore \cos^2\theta = 1 - \sin^2\beta \operatorname{cosec}^2\phi$$
$$= 1 - \sin^2\beta - \sin^2\beta \cot^2\phi = \cos^2\beta - \sin^2\alpha = \&c.$$

12.



Let $2\psi = 2QPC$ be the angle under which the circle is seen from P, when it has revolved round O, through an angle θ, from its initial position, when CO was coincident with OP;

then $\sin^2\psi = \cos^2 QCP = \dfrac{r^2}{PC^2}$

$$= \frac{r^2}{OP^2 + OC^2 + 2\,OP.OC\cos\theta}:$$

hence ψ is greatest, when $\cos\theta$ is least, and $= -1$,

$$\text{so that } \sin\alpha = \frac{r}{OP - OC},$$

and ψ is least, when $\cos\theta$ is greatest, and $= +1$,

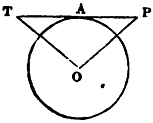
$$\text{so that } \sin\beta = \frac{r}{OP + OC};$$

$$\therefore \frac{OP + OC}{OP - OC} = \frac{\sin\alpha}{\sin\beta},$$

and $\dfrac{OC}{OP} = \dfrac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = (\text{by question})\ \dfrac{\tan^2(\frac{1}{4}\pi + \frac{1}{2}\phi) - 1}{\tan^2(\frac{1}{4}\pi + \frac{1}{2}\phi) + 1}$

$$= \frac{\sin^2 - \cos^2}{\sin^2 + \cos^2} = -\cos(\tfrac{1}{2}\pi + \phi) = \sin\phi.$$

13.



By (93 Ex. 4) $d^2 = 2rh$: let T be the top of the mast, P the peak, h, h', the vertical heights of peak and mast; then distance required $= PT = PA + AT$, A being the point where PT touches the earth's surface:

hence $PT = \sqrt{(2rh)} + \sqrt{(2rh')} = \sqrt{20000} + \sqrt{121.2121}$ &c., (*since* $r = 4000$ *miles*, $h = 2\frac{1}{2}$ *miles*, and $h' = 80$ feet $= \frac{1}{66}$ mile)

$$= 152\tfrac{1}{2} \text{ miles}.$$

314. By (118) $A = nr^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n}$, $B = nr^2 \sin\frac{\pi}{n} \div \cos\frac{\pi}{n}$;

$$\text{also } A' = 2nr^2 \sin\frac{\pi}{2n}\cos\frac{\pi}{2n} = nr^2 \sin\frac{\pi}{n},$$

$$B' = 2nr^2 \sin\frac{\pi}{2n} \div \cos\frac{\pi}{2n};$$

$$\text{hence } AB = n^2r^4\sin^2\frac{\pi}{n} = A'^2, \text{ or } A' = \sqrt{(AB)};$$

$$\text{and } \frac{A}{B} = \cos^2\frac{\pi}{n}, \quad \frac{A'}{B'} = \cos^2\frac{\pi}{2n} = \tfrac{1}{2}\left(1 + \cos\frac{\pi}{n}\right) = \tfrac{1}{2}\left(1 + \sqrt{\frac{A}{B}}\right);$$

$$\therefore B' = \&\text{c.}$$

315. Let a be a side of the equilateral triangle;
then its area $= \frac{1}{2}a^2\cos 60° = \frac{1}{4}a^2\sqrt{3}$, and its perimeter $= 3a$:
hence $a-d$, a, $a+d$, may represent the sides of the other triangle,
and its area $= \sqrt{\{\frac{3}{2}a(\frac{1}{2}a+d)\frac{1}{2}a(\frac{1}{2}a-d)\}}$
$= \frac{1}{4}a\sqrt{\{3(a^2-4d^2)\}} =$ by question $\frac{3}{20}a^2\sqrt{3}$,
whence $d = \frac{2}{5}a$, and the sides are $\frac{3}{5}a$, a, $\frac{7}{5}a$;
that is, they are as 3, 5, 7:

$$\therefore \text{ cosine of largest angle} = \frac{9+25-49}{30} = -\tfrac{1}{2} = \cos 120°.$$

316. $\cos A' = \dfrac{b'^2 + c'^2 - a'^2}{2b'c'} = \dfrac{b^2\cos^2 B + c^2\cos^2 C - a^2\cos^2 A}{2bc\cos B\cos C}$

$$= \frac{\sin^2 B\cos^2 B + \sin^2 C\cos^2 C - \sin^2 A\cos^2 A}{2\sin B\sin C\cos B\cos C} \text{ by (110)}$$

$$= \frac{\sin^2 2B + \sin^2 2C - \sin^2 2A}{2\sin 2B\sin 2C}$$

$$= \frac{\sin^2 2B - \sin(2A+2C)\sin(2A-2C)}{2\sin 2B\sin 2C}$$

$$= \frac{-\sin(2A+2C) + \sin(2A-2C)}{2\sin 2C}$$

(*since* $\sin(2A+2C) = \sin(360° - 2B) = -\sin 2B$)

$= -\cos 2A$; $\therefore A' = 180° - 2A$.

317. *From* (1) $\sin^2\theta = \dfrac{m-b}{a-b}$, $\cos^2\theta = \dfrac{a-m}{a-b}$, $\tan^2\theta = \dfrac{m-b}{a-m}$;

from (2), $\tan^2\phi = \dfrac{a-n}{n-b}$;

$\therefore$ (3) gives $a^2(m-b)(n-b) = b^2(m-a)(n-a)$,
or $a^2\{mn-(m+n)b+b^2\} = b^2\{mn-(m+n)a+a^2\}$;
$\therefore (a^2-b^2)mn = (m+n)ab(a-b)$, or $(a+b)mn = (m+n)ab$.

18. AB (the staff) $= 1$, BC (the tower) $= 200$, $CD = 100$:
then $\tan BDC = 2$, $\tan ADC = 2.01$;

and $\tan ADB = \tan(ADC - BDC) = \dfrac{2.01-2}{1+4.02}$

$$= \tfrac{1}{502} = \theta \text{ for } \angle ADB;$$

$\therefore$ no. of seconds in $ADB = \frac{206265}{502} = 410.8$ or $411''$ nearly
$= 6'51''$.

19. By (122), $\tan\frac{1}{2}(B-C) = \dfrac{b-c}{b+c}\tan\frac{1}{2}(B+C)$,

or $\tan\dfrac{B-180^\circ+A+B}{2} = -\cot(B+\frac{1}{2}A) = \dfrac{b-c}{b+c}\cot\frac{1}{2}A$,

or $\tan(B+\frac{1}{2}A) = \dfrac{c+b}{c-b}\tan\frac{1}{2}A =$ (in this case) $\frac{10}{4}\tan\frac{1}{2}A$;

$\therefore L\tan(B+\frac{1}{2}A) = 1 - 2\log 2 + L\tan 3^\circ\,18'\,42'' = 9.1603480$;
and D for $10'' = 1486$, $d = 397$; $\therefore n = \frac{397}{1486}\times 10 = 2.67$;
and $\therefore B+\frac{1}{2}A = 8^\circ\,13'\,52''.67$,
or $B = 4^\circ\,55'\,10''.67 = 4^\circ\,55'\,11''$ nearly,
and $C = 180^\circ - (A+B)$.

20. $\tan\theta$, $\tan\phi$, $\tan\psi$, are plainly roots of
$$x^3 - (1+\tfrac{4}{3}\sqrt{3})x^2 + (1+\tfrac{4}{3}\sqrt{3})x - 1 = 0,$$
whence $x^3 - 1 = (1+\frac{4}{3}\sqrt{3})(x^2-x)$;
$\therefore x-1 = 0$, and $x^2+x+1 = (1+\frac{4}{3}\sqrt{3})x$;
$\therefore$ (i) $x = 1$, and (ii) $x^2 - \frac{4}{3}\sqrt{3}x + \frac{4}{3} = \frac{1}{3}$,

or $x = \dfrac{2\pm 1}{\sqrt{3}} = \sqrt{3}$ or $\frac{1}{3}\sqrt{3}$:

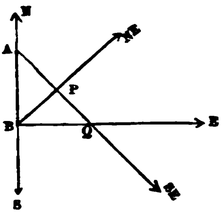
$\therefore \theta = 45^\circ$ or $n\pi + \frac{1}{4}\pi$, $\phi = 60^\circ$ or $n\pi + \frac{1}{3}\pi$,
$\psi = 30^\circ$ or $n\pi + \frac{1}{6}\pi$.

21. We have here to show that $\frac{1}{2}(\alpha+\beta) = \phi$:

now $\tan\frac{1}{2}(\alpha+\beta) = \dfrac{\tan\frac{1}{2}\alpha + \tan\frac{1}{2}\beta}{1-\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta} = \dfrac{\tan^3\frac{1}{2}\beta + \tan\frac{1}{2}\beta}{1-\tan^4\frac{1}{2}\beta}$

$$= \frac{\tan\frac{1}{2}\beta}{1-\tan^2\frac{1}{2}\beta} = \tfrac{1}{2}\tan\beta = \tan\phi.$$

322.



A, B, the two points of observation, P, Q, the objects;
then $AB=8$, $\angle BAQ=45°$, $\angle ABP=45°$;
$\therefore BP$ is perp. to AQ,
and $PQ=BQ\cos 45°$
$=AB\cos 45°=4\sqrt{2}$
$=5.656852$ *miles*
$=5$ *miles* 1156 *yds.*

323. By (85) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$,
or $(1+x+x^2)\tan A=x^3\tan^3 A$, whence $x^3\tan^2 A=1+x+x^2$,
and $\tan^2 C=x^4\tan^2 A=x(1+x+x^2)$;
$\therefore x^3\sec^2 A=1+x+x^2+x^3=\sec^2 C$;
and $\sin 2A=2\sin A\cos A=2\tan A\div\sec^2 A$
$=2x^3\tan A\div x^3\sec^2 A=2x\tan C\div\sec^2 C$
$=x\sin 2C$.

324. $\cos^{-1}\frac{1}{a}+\cos^{-1}\frac{a}{x}=\cos^{-1}\frac{1}{b}+\cos^{-1}\frac{b}{x}$, or $\alpha+\theta=\beta+\phi$, suppose;

$\therefore \cos(\alpha+\theta)=\cos(\beta+\phi)$,

or $\frac{1}{a}\times\frac{a}{x}-\sqrt{\left\{\left(1-\frac{1}{a^2}\right)\left(1-\frac{a^2}{x^2}\right)\right\}}$

$=\frac{1}{b}\times\frac{b}{a}-\sqrt{\left\{\left(1-\frac{1}{b^2}\right)\left(1-\frac{b^2}{x^2}\right)\right\}}$;

$\therefore \left(1-\frac{1}{a^2}\right)\left(1-\frac{a^2}{x^2}\right)=\left(1-\frac{1}{b^2}\right)\left(1-\frac{b^2}{x^2}\right)$, or $x^2=a^2b^2$.

325. Let AB, CD, be any two sides, and join AC, BD, crossing each other in P;
then, joining AD, $\angle APB=\angle ADP+\angle PAD$
$=\angle ADB+\angle CAD$
$=2(\angle$ at circumference on $AB)=\frac{2\pi}{n}$:

hence P is a point in a segment of a circle, described on base AB, containing an angle $\frac{2\pi}{n}$,

and (114) the diameter of this circle is $\frac{AB}{\sin P}=a\operatorname{cosec}\frac{2\pi}{n}$.

326. Given A, a, b, and $a\angle b$: let $AB=c$, $AB'=c'$, be the two values found for the third side; draw CD perp. to ABB': then $b\cos A = AD = \frac{1}{2}(c+c')$, $b\sin A = CD = \frac{1}{2}(c'-c)\tan B$:

$$\therefore\ 4b^2(\cos^2 A + \sin^2 A) = 4b^2 = (c+c')^2 + (c'-c)^2\tan^2 B$$
$$= (c^2+c'^2)\sec^2 B + 2cc'(1-\tan^2 B);$$
$$\therefore\ c^2 + c'^2 + 2cc'\cos 2B = 4b^2\cos^2 B.$$

27. By (115) $rr_1r_2r_3 = \dfrac{S^4}{s(s-a)(s-b)(s-c)} = S^2$;

and $r_1 + r_2 + r_3 - r = S\left\{\dfrac{1}{s-a} + \dfrac{1}{s-b} + \dfrac{1}{s-c} - \dfrac{1}{s}\right\}$

$$= S\left\{\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right\}$$

$$= \frac{c}{S}\{s(s-c) + (s-a)(s-b)\} = \frac{abc}{S} = 4R, \text{ by (114).}$$

28. Suppose that the angular points of r regular figures, each of n sides, meet at a point;

then $r\dfrac{n-2}{n}180^\circ = 360^\circ$, by Euc. i. 15. Cor. 2;

$\therefore\ r = \dfrac{2n}{n-2} = 2 + \dfrac{4}{n-2}$, which must, therefore, be an integer:

hence $n-2 = 1, 2$, or 4, and $n = 3, 4$, or 6.

29. $\tan(\theta-\phi) = \frac{1}{2}(\tan\theta - \tan\phi)$, or $\dfrac{\sin(\theta-\phi)}{\cos(\theta-\phi)} = \dfrac{\sin(\theta-\phi)}{2\cos\theta\cos\phi}$;

$\therefore$ (i) $\sin(\theta-\phi) = 0$, or (ii) $\cos(\theta-\phi) = 2\cos\theta\cos\phi$:
(i) gives $\theta - \phi = m\pi$, and $\theta = m\pi + \phi$;
$\therefore\ \sin(\theta+\phi) = \tan(\theta-\phi)$ gives $\sin(m\pi + 2\phi) = \tan m\pi = 0$,
or $m\pi + 2\phi = n\pi$,
whence $\phi = (n-m)\frac{1}{2}\pi$, and $\theta = m\pi + \phi = (n+m)\frac{1}{2}\pi$:
(ii) gives $\cos\theta\cos\phi + \sin\theta\sin\phi = 2\cos\theta\cos\phi$,
or $\tan\theta = \cot\phi = \tan(\frac{1}{2}\pi - \phi)$, whence $\theta = m\pi + (\frac{1}{2}\pi - \phi)$;
and $\tan(\theta-\phi) = \sin(\theta+\phi) = \sin(m\pi + \frac{1}{2}\pi) = (-1)^m$;
$\therefore\ \theta - \phi = n\pi + (-1)^m\frac{1}{4}\pi$, and $\theta + \phi = m\pi + \frac{1}{2}\pi$,
whence θ and ϕ.

30. Let $\operatorname{vers}\theta = 1 + x$, $\operatorname{vers}\phi = 1 - x$;
then $\cos\theta = -x$, $\cos\phi = x$, or $\theta = \pi \pm \phi$:
hence $2\sqrt{(1-x^2)} = \tan(\theta-\phi) = \tan\pi$ or $\tan(\pi - 2\phi)$;
$\therefore$ (i) $2\sqrt{(1-x^2)} = 0$, or $x = \pm 1$;

(ii) $2\sqrt{(1-x^2)}=-\tan 2\phi=\frac{2\tan\phi}{\tan^2\phi-1}=\frac{2x\sqrt{(1-x^2)}}{1-2x^2}$;

$\therefore\ \sqrt{(1-x^2)}=0$, or $x=\pm 1$, as before;
and $1-2x^2=x$, whence $x=-1$ or $\frac{1}{2}$.

331. By (Euc. vi. 3) $\frac{b}{c}=\frac{m}{n}=\frac{\sin B}{\sin(A+B)}$,

whence $\tan B=\frac{m\sin A}{n-m\cos A}$: put $n=k\cos\phi\sin A$, $m=k\sin\phi$;

then $\tan B=\frac{m\sin A}{k\sin(A-\phi)}=\frac{\sin A\sin\phi}{\sin(A-\phi)}$,

where $\tan\phi=\frac{k\sin\phi}{k\cos\phi}=\frac{m}{n}\sin A$.

332. Let $C=\pi-\theta$;

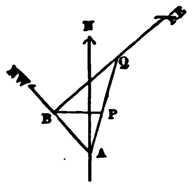
then $\cos\frac{1}{2}C=\sqrt{\frac{s(s-c)}{ab}}=\frac{1}{2}\sqrt{\frac{(a+b+c)(a+b-c)}{ab}}$

$=\sin\frac{1}{2}\theta=\frac{1}{2}\theta$, nearly;

$\therefore\ \theta=\sqrt{\frac{(a+b+c)(a+b-c)}{ab}}=\sqrt{\frac{2(a+b)(a+b-c)}{ab}}$, nearly,

since $c^2=a^2+b^2-2ab\cos C=a^2+b^2+2ab\cos\theta$
$=a^2+b^2+2ab(1-\frac{1}{2}\theta^2\ \&c.)=(a+b)^2$, nearly:
$\therefore$ the value of θ in *seconds* = &c. (29).

333.



$\angle NAQ=15°$, $\angle NAB=45°$,
$\angle QBP=45°$, $AB=5$;
$\therefore\ \angle ABP=45°$,
$\angle BAP=45°+15°=60°$,
$\angle APB=75°$, $\angle ABQ=90°$;
hence $AQ=AB\sec QAB$
$=5\sec 60°=10$;

and $AP=AB\frac{\sin 45°}{\sin 75°}=5(\sqrt{3}-1)$;

hence $PQ=15-5\sqrt{3}=6.34$ *miles*.

334. P the centre of inscribed circle, $\angle PAB=\phi$, PD perp. to AB:
then by (115. N.B.) $AD=s-a=2c-a=r\cos\phi$;
$\therefore\ a=2c-r\cos\phi$

and $\sin^2\phi=\sin^2\frac{1}{2}A=\frac{(s-b)(s-c)}{bc}=\frac{2c-b}{b}$, whence $b = \dfrac{2c}{1+\sin^2\phi}$;

$\therefore\ a+b = 2s - c = 3c = (2c - r\cos\phi) + \dfrac{2c}{1+\sin^2\phi}$,

or $r(1+\sin^2\phi) = c\cos\phi$.

35. $$\sin^{-1}\frac{\sin z}{1+n} = \sin^{-1}\{\sin z(1-n)\} = \tan^{-1}\frac{(1-n)\sin z}{\sqrt{\{1-(1-n)^2\sin^2 z\}}}$$

$$= \tan^{-1}\frac{(1-n)\sin z}{\sqrt{(\cos^2 z + 2n\sin^2 z)}}, \text{ nearly}$$

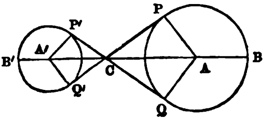
$$= \tan^{-1}\frac{(1-n)\tan z}{\sqrt{(1+2n\tan^2 z)}}$$

$$= \tan^{-1}\{(1-n)\tan z(1-n\tan^2 z + \&c.)\}$$

$$= \tan^{-1}\{(1-n\sec^2 z)\tan z\} = \tan^{-1}x, \text{ suppose:}$$

$\therefore\ r = \tan\tan^{-1}x = x$.

36.



Let $\angle PAC = \theta$;

then $a = AA' = AC + A'C$

$= R\sec\theta + r\sec\theta = c\sec\theta$,

or $\cos\theta = \dfrac{c}{a}$;

and string $PBQ = 2\pi R - 2\theta R$,

string $P'B'Q' = 2\pi r - 2\theta r$,

string $PQ' = P'Q = PC + CQ' = AC\sin\theta + A'C\sin\theta$

$= a\sin\theta = \sqrt{(a^2-c^2)}$;

hence whole string $= 2\sqrt{(a^2-c^2)} + 2(R+r)(\pi-\theta)$

$$= 2\left\{\sqrt{(a^2-c^2)} + c\cos^{-1}\left(-\frac{c}{a}\right)\right\}.$$

37. $$a = \frac{1}{\cos\phi} + \frac{\sin^2\theta}{\cos^3\theta}\cdot\frac{1+\sin^2\theta}{\sin^2\theta} = \frac{\cos^2\theta + 1 + \sin^2\theta}{\cos^3\theta} = 2\sec^3\theta;$$

$$b = \frac{\sin\theta}{\cos\theta} - \frac{\sin^3\theta}{\cos^3\theta}\cdot\frac{1+\sin^2\theta}{\sin^2\theta}$$

$$= \frac{(\sin\theta\cos^2\theta - 1 - \sin^2\theta)}{\cos^3\theta} = -2\tan^3\theta;$$

but $\sec^2\theta = 1 + \tan^2\theta$;

$\therefore\ (\tfrac{1}{2}a)^{\frac{2}{3}} = 1 + (\tfrac{1}{2}b)^{\frac{2}{3}}$, or $a^{\frac{2}{3}} - b^{\frac{2}{3}} = 2^{\frac{2}{3}}$.

38. A, B, the two positions of the spectator, PQ, $P'Q'$, the towers, so that $P'PA$ is in one line, and $\angle PBQ = 2\angle P'BQ'$; $AQ = a$, $AQ' = b$, $AB = c$, $PQ = h$, $P'Q' = h'$:

then $\frac{h}{a}=\frac{h'}{b}$, and $\tan PBQ=\frac{h}{a-c}$, $\tan P'BQ'=\frac{h'}{b-c}$;

$\therefore$ since $\tan PBQ=\tan 2P'BQ'$, we have $\frac{h}{a-c}=\frac{\frac{2h'}{b-c}}{1-\frac{h'^2}{(b-c}}$

or $1-\frac{h'^2}{(b-c)^2}=\frac{2h'}{h}\cdot\frac{a-c}{b-c}=\frac{2b}{a}\cdot\frac{a-c}{b-c}$;

$\therefore$ $h'=$ &c., and $h=\frac{a}{b}h'$.

339. Let AC, AD, cut BE in F, G;
draw AH bisecting FG at right angles:
then $FG=2FH=2AF\sin 18^\circ=AB\sec 36^\circ\sin 18^\circ$,
since $AB=AF\cos BAF+BF\cos ABF=2AF\cos 36^\circ$
$\therefore$ area of figure on AB: area of figure on $FG=AB^2$:
$AB^2\sec^2 36^\circ\sin^2 18^\circ=\sin^2 54^\circ:\ \sin^2 18^\circ=3+\sqrt{5}:\ 3-\sqrt{5}$

340. Draw CC' bisecting AB in F:
then triangle $CBC'=BF\,.\,CF=BC\cos 60^\circ\,.\,BC\sin 60$
$=\frac{1}{2}a^2\sin 120^\circ=\frac{1}{4}a^2\sqrt{3}$;
and sector $BCAC'=\frac{1}{2}\theta r^2=\frac{1}{3}\pi a^2$;
$\therefore$ area $CAC'=\frac{1}{12}a^2(4\pi-3\sqrt{3})=$ area CBC'.

341. The angle of the sector being 2,

the area $=\frac{2}{2\pi}\times\pi r^2=r^2=4$ *sq. in.*;

and the arc of the sector forms the circumference of the b of the cone; but this arc $=2r$ (since $\theta=2$) $=4$ *in.*, which $=2\pi r'$, if r' be the radius of the base;

hence $r'=\frac{2}{\pi}$, and area $=\pi r'^2=\frac{4}{\pi}=1\frac{3}{11}$ *sq. in.*

342. $\tan^2 x=\frac{\tan(x-a)}{\tan x}$;

$$\therefore\ \cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}=\frac{\tan x-\tan(x-a)}{\tan x+\tan(x-a)}=\frac{\sin a}{\sin(2x-a)};$$

hence $2\sin a=2\cos 2x\sin(2x-a)=\sin(4x-a)-\sin a$,
or $3\sin a=\sin(4x-a)$, which cannot be greater than 1.

343. $7-\frac{10}{3}\sqrt{3}=\frac{1-\cos 2x}{1+\cos 2x}+\sec 2x=\frac{\sec 2x-1}{\sec 2x+1}+\sec 2x$

$=(\sec 2x+1)-\frac{2}{\sec 2x+1}=z-\frac{2}{z}$, suppose:

$\therefore z^2-(7-\frac{10}{3}\sqrt{3})z-2=0$, whence $z=1+\frac{2}{3}\sqrt{3}$ or $6-4\sqrt{3}$;
$\therefore \sec 2x=\frac{2}{3}\sqrt{3}$, (the other root being impossible,) and $2x=30°$, or, generally, $=2n\pi\pm\frac{1}{6}\pi$.

44. Area $APB=\frac{1}{2}$ (semicircle) $=\frac{1}{4}\pi r^2$
$=$ triangle $APC+$ sector PCB
$=\frac{1}{2}r^2\sin ACP+\frac{1}{2}r^2$ (circ. meas. of $\angle BCP$)
$=\frac{1}{2}r^2(\cos\theta+\frac{1}{2}\pi-\theta)$; $\therefore \cos\theta=\theta$.

45. A, B, the centres of given circles, C the middle point of AB; E the centre of circle, radius r', which touches AB and the two former circles, F the centre of circle, radius r'', which touches the three; then $AC=\frac{1}{2}r$, $AF^2=AC^2+CF^2$,
or $(r-r'')^2=\frac{1}{4}r^2+(2r'+r'')^2$; (1)
again, $AD^2=AC^2+CD^2$, or $(r-r')^2=\frac{1}{4}r^2+r'^2$; (2)
from (2), $r'=\frac{3}{8}r$, and then from (1), $r''=\frac{3}{56}r$.

46. $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\dfrac{x-y}{1+xy}$;

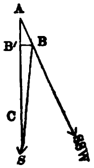
$\tan^{-1}\dfrac{1}{2y}-\tan^{-1}\dfrac{1}{2x}=\tan^{-1}\dfrac{2x-2y}{1+4xy}$.

$\therefore \dfrac{x-y}{1+xy}=\dfrac{2(x-y)}{1+4xy}$, whence (i) $x-y=0$, (ii) $xy=\frac{1}{2}$:
(i) is impossible, by the latter part of the question:
but (ii) gives $\tan^{-1}x-\tan^{-1}\dfrac{1}{2x}=\tan^{-1}\dfrac{2x^2-1}{3x}=\dfrac{\pi}{4}=\tan^{-1}$;
$\therefore 2x^2-1=3x$, and $x=\frac{1}{4}(3\pm\sqrt{17})$, $y=$ &c.

47. $\tan 2\theta-\tan\theta=\dfrac{\sin(2\theta-\theta)}{\cos 2\theta\cos\theta}=\dfrac{\sin\theta}{\cos 2\theta\cos\theta}$

$=\dfrac{\theta-\frac{1}{6}\theta^3}{(1-2\theta^2)(1-\frac{1}{2}\theta^2)}+$ terms in θ^4, &c.
$=(\theta-\frac{1}{6}\theta^3)(1+2\theta^2)(1+\frac{1}{2}\theta^2)+$ &c.
$=(\theta-\frac{1}{6}\theta^3)(1+\frac{5}{2}\theta^2)+$ &c. $=\theta+\frac{7}{3}\theta^3+$ &c.:
hence $u=\frac{1}{3}\{2(2\theta-\frac{4}{3}\theta^3+\text{&c.}-\theta+\frac{1}{6}\theta^3-\text{&c.})+\theta+\frac{7}{3}\theta^3+\text{&c.}\}$
$=\frac{1}{3}\{2(\theta-\frac{7}{6}\theta^3)+\theta+\frac{7}{3}\theta^3\}+$ &c.
$=\theta+$ terms in θ^4, &c. $=\theta$, nearly.

48.



AB the first ship, which is sailing at an angle of $22\frac{1}{2}°$ to the direction of the other ship C;
$\therefore$ drawing BB' perp. to AS, we have (θ being the circ. meas. of $20'$) $BB'=CB\cdot\theta$ nearly,
or $AB\sin A=150\sin 22\frac{1}{2}°=d\theta$;
$\therefore d=$ (in *feet*) $\dfrac{75\sqrt{(2-2\cos 45°)}}{\theta}$

$=$ [illegible]

$=$ [illegible]

349. [illegible] $\sin C$:

$\therefore$ [illegible]

[illegible]

$$= \text{[illegible]}$$

$\cos C =$ [illegible]

$$= \text{[illegible]};$$

$\therefore \dfrac{\cos C}{\cos B} = \dfrac{x^2-1}{x^2-1} \times \dfrac{1-x^4}{2x^2}$, and $\dfrac{\tan B}{\tan C} = \dfrac{x^2-1}{x^2+1}$.

350. $$\tan a \tan x = \frac{\sin^2(a+x)\cos^2(a-x) - \cos^2(a+x)\sin^2(a-x)}{\cos^2(a-x)\cos^2(a-x)}$$

$$= \frac{\sin\{(a+x)+(a-x)\} \times \sin\{(a+x)-(a-x)\}}{\cos^2(a-x)\cos^2(a-x)}$$

$$= \frac{\sin 2a \sin 2x}{\cos^2(a+x)\cos^2(a-x)};$$

$\therefore$ (i) $\sin x = 0$.

and (ii) $\cos(a+x)\cos(a-x)$, or $\cos^2 x - \sin^2 a = \pm 2\cos a \cos x$:
(i) gives $x = n\pi$, and $\cos x = \pm 1$: (ii) gives $(\cos x \mp \cos a)^2 = 1$,
or $\cos x = \pm \cos a \pm 1 = \pm 2\cos^2 \frac{1}{2}a$ or $\pm 2\sin^2 \frac{1}{2}a$.

351. OA, OB, the bounding radii of the quadrant, OC the bisecting radius; CD = radius of circle required, whose centre D will be in OC: bisect OA in E, which will be the centre of circle on OA; then DE will pass through the point of contact of circle required with circle on OA;
and, if $OA = a$, $CD = r$,
we have $DE^2 = OE^2 + OD^2 - 2\,OE \, . \, OD \cos 45°$,
or $(\frac{1}{2}a + r)^2 = \frac{1}{4}a^2 + (a-r)^2 - \frac{1}{2}a(a-r)\sqrt{2}$,

whence $\dfrac{r}{a} = \dfrac{\sqrt{2}-1}{3\sqrt{2}-1}$.

52. The sides of the second triangle (115 N. B.) are

$$a' = s - a,\ b' = s - b,\ c' = s - c;$$

now $r' = \dfrac{a'b'c'}{4S'}$, $r'' = \dfrac{S'}{s'}$;

$$\therefore\ r'r'' = \frac{a'b'c'}{4s'} = \frac{(s-a)(s-b)(s-c)}{2s} = \frac{S^2}{2s^2} = \tfrac{1}{2}r^2.$$

53, Area DAB (109) $= \tfrac{1}{2}AB^2\dfrac{\sin A \sin B}{\sin D} = \tfrac{1}{2}a^2\dfrac{\sin 72° \sin 72°}{\sin 36°}$

$$= a^2 \cos 36° \cos 18°;$$

and $ABCDE$ (117) $= \tfrac{5}{4}a^2 \cot 36°$;
and the ratio of these is $4 \sin 36° \cos 18° : 5 = \sqrt{5} : 1$.

54. Let $\operatorname{cosec}\theta = \sqrt{(x^2 - 2x + 2)}$; then $\cot\theta = x - 1$:

let $\operatorname{chd}\phi = \dfrac{2}{\sqrt{(x^2 + 2x + 2)}} = 2\sin\tfrac{1}{2}\phi$;

then $\operatorname{cosec}\tfrac{1}{2}\phi = \sqrt{(x^2 + 2x + 2)}$, and $\cot\tfrac{1}{2}\phi = x + 1$.
Now, by question, $2\theta = \tfrac{1}{6}\pi + \phi$, or $\cot(\theta - \tfrac{1}{2}\phi) = \cot\tfrac{1}{12}\pi$:

$$\therefore\ \frac{\cot\theta \cot\tfrac{1}{2}\phi + 1}{\cot\tfrac{1}{2}\phi - \cot\theta} = \frac{x^2}{2} = \cot 15° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{2};$$

$\therefore\ x = \pm(\sqrt{3} + 1)$.
Other values might be obtained by taking $\cot\theta = 1 - x$, or $\cot\tfrac{1}{2}\phi = -(x + 1)$.

55. Let D, E, F, be the angles of the hexagon, opposite to the angles A, B, C: then, in figure $ABDC$, $\angle\angle ABD$, ACD, are right angles; hence a circle may be described round $ABDC$, and this must be the same as that round triangle ABC, which is part of figure $ABDC$: similarly for points E and F.

Also area BDC (109) $= \tfrac{1}{4}BC^2\dfrac{\sin DBC \sin DCB}{\sin D}$

$$= \tfrac{1}{2}a^2\frac{\cos B \cos C}{\sin A};$$

and area $ABC = \tfrac{1}{2}a^2\dfrac{\sin B \sin C}{\sin A}$;

$\therefore$ area $BDC = \cot B \cot C \times$ area ABC,
and area BDC + area AEC + area AFB
$= (\cot B \cot C + \cot A \cot C + \cot A \cot B)$ area ABC
$=$ (Ex. 15. 2) area ABC; $\therefore$ whole hexagon $=$ 2 area ABC

356. $\frac{a}{b}=\frac{\sin A}{\sin B}=\frac{\sin \alpha}{\sin \beta}$, if α, β, be the circular measures of A and B;

$\therefore$ since $a=$ (nearly) b, we have $\alpha=$ (nearly) β or $\pi-\beta$,
$=$ (suppose) $\beta+\delta$ or $\pi-(\beta+\delta)$, where δ is small:

$$\text{hence } \frac{a}{b}=\frac{\sin(\beta+\delta)}{\sin\beta}=\cos\delta+\sin\delta\cot\beta$$

$$=1+\delta\cot\beta, \text{ nearly,}$$

$$\text{or } \delta=\frac{a-b}{b}\tan\beta,$$

and $d=$ value of δ in seconds $=\frac{a-b}{b\sin 1''}\tan B$, by (95):

but $A+B+C=180°$; $\therefore$ (i) if $\alpha=\beta+\delta$, or $A=B+d$,
then $2B+d+C=180°$, or $C=(180°-2B)-d$

$$=\text{(in seconds) } (180°-2B)\times 60\times 60-\frac{a-b}{b\sin 1''}\tan B;$$

(ii) if $\alpha=\pi-(\beta+\delta)$, or $A=180°-(B+d)$,
then $180°-d+C=180°$, or $C=d=$ &c.

357. $\sin(\theta-\phi)=\frac{1}{\sqrt{10}}$; $\frac{\sin^2\theta\cos^2\phi}{\cos^2\theta\sin^2\phi}=\frac{\tan^2\theta}{\tan^2\phi}=\frac{5+3}{5-3}=4$;

$$\therefore \frac{\tan\theta}{\tan\phi}=\pm 2;$$

$$\text{but } \tan(\theta-\phi)=\frac{\sin(\theta-\phi)}{\sqrt{(1-\sin^2(\theta-\phi))}}=\pm\tfrac{1}{3}=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$$

$$=\frac{2\tan\phi-\tan\phi}{1+2\tan^2\phi},$$

whence $\tan\phi=\pm 1$ or $\pm\frac{1}{2}$, and $\tan\theta=\pm 2$ or ± 1.
There are also other values by putting $\tan\theta=-2\tan\phi$.

358. Let $B'=B+\beta$, $C'=C+\gamma$, be the new values of B and C, when A becomes $A+\alpha$;

$$\therefore \sin B=\sin(A+C)=\frac{b}{c}\sin C,\ \sin(A'+C')=\frac{b}{c}\sin C',$$

since $A'+B'+C'=180°$;

$$\therefore \sin(A'+C')-\sin(A+C)=\frac{b}{c}(\sin C'-\sin C),$$

or $\sin\frac{1}{2}(A'-A+C'-C)\cos\frac{1}{2}(A'+A+C'+C)$
$=\frac{b}{c}\sin\frac{1}{2}(C'-C)\cos\frac{1}{2}(C'+C)$,
that is, $\sin\frac{1}{2}(\alpha+\gamma)\cos\{A+C+\frac{1}{2}(\alpha+\gamma)\}$
$=\frac{b}{c}\sin\frac{1}{2}\gamma\cos(C+\frac{1}{2}\gamma)$, or $\frac{1}{2}(\alpha+\gamma)\cos(A+C)=\frac{b}{c}\times\frac{1}{2}\gamma\cos C$,
to the first power of small quantities,
that is, $-(\alpha+\gamma)\cos B=\gamma\frac{\sin B}{\sin C}\cos C$;
$\therefore\ \gamma=-\frac{\sin C\cos B}{\sin(B+C)}d=$ &c.; and similarly for β.

59. Take a circle, centre C, diameter AB:
take $\angle BAP=\theta$, and draw PMP' perp. on AB:
then $2\sin\theta=\frac{2BP}{AB}=2\sqrt{\frac{BM}{AB}}$ (Euc. vi. 8) $=\sqrt{\frac{2(BC-CM)}{BC}}$

$$=\sqrt{\left(2-\frac{2CM}{CP}\right)}=\pm\sqrt{(2-2\cos\theta)}:$$

the two signs correspond to the angles BAP, BAP', for which the sines are equal in magnitude, but differ in sign, while $\cos 2\theta$ remains the same for both,
since $\cos BCP=\cos BCP'$.
For (ii) see (308): let CP revolving come to CP',
so that $\angle ACP'=\angle BCP$;
then $\sin BCP'$, or $\sin 2\theta'$, has the same value as $\sin 2\theta$,
but $\sin\theta'$ becomes $\sin\frac{1}{2}BCP'=\sin\frac{1}{2}(\pi-2\theta)=+\cos\theta$;
if now CP go on revolving till it come again to the positions CP, CP', then the values of $\sin 2\theta''$, $\sin 2\theta'''$ will still be the same as $\sin 2\theta$, but $\sin\theta''=\sin\frac{1}{2}(2\pi+2\theta)=-\sin\theta$,
and $\sin\theta'''=\sin\frac{1}{2}(3\pi-2\theta)=-\cos\theta$.

60. Here $3\theta+4\theta+5\theta=180°$;
$\therefore\ \theta=15°$, and the angles are $45°$, $60°$, $75°$;
now (115) area $=rs=$ (see 76) $s^2\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C$;
$\therefore\ s=r\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$,
and area $=r^2\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$

$$=r^2\sqrt{\frac{1+\cos 45°}{1-\cos 45°}}\times\sqrt{3}\times\sqrt{\frac{1+\cos 75°}{1-\cos 75°}}$$

$$=r^2(\sqrt{2}+1)\sqrt{3}\sqrt{\frac{2\sqrt{2}+\sqrt{3}-1}{2\sqrt{2}-\sqrt{3}+1}}$$

$$=r^2(\sqrt{2}+1)\sqrt{3}\sqrt{\frac{(2\sqrt{2}+\sqrt{3})^2-1}{(2\sqrt{2}+1)^2-3}}$$

$$=r^2(\sqrt{2}+1)\sqrt{3}\sqrt{\frac{5+2\sqrt{6}}{3+2\sqrt{2}}}$$

$$=r^2(\sqrt{2}+1)\sqrt{3}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+1}=r^2(3+\sqrt{6}).$$

361. Put $k\cos\phi=\sqrt{(1+\tan^2\alpha\tan^2\beta)}$,
and $k\sin\phi=\sqrt{(1-\tan^2\alpha\tan^2\beta)}$;

then $k^2=2$, and $\tan^2\phi=\dfrac{1-\tan^2\alpha\tan^2\beta}{1+\tan^2\alpha\tan^2\beta}$,

or $\cos 2\phi=\dfrac{1-\tan^2\phi}{1+\tan^2\phi}=\tan^2\alpha\tan^2\beta$;

$\therefore\ k\sin(\theta+\phi)=\tan\alpha+\tan\beta=\sin(\alpha+\beta)\sec\alpha\sec\beta.$

362. Let D, E, F, be the bisections of BC, CA, AB:

then $R_1(114)=\dfrac{BC}{2\sin F}=\dfrac{a}{2\sin F}$, $R_2=\dfrac{b}{2\sin F}$,

$R_3=\dfrac{c}{2\sin E}$, $R_4=\dfrac{a}{2\sin E}$, $R_5=\dfrac{b}{2\sin D}$, $R_6=\dfrac{c}{2\sin D}$;

$\therefore R_1R_3R_5=R_2R_4R_6.$

Also let $S=$ area of each of the six triangles $=\frac{1}{6}$ area ABC

$AD=p$, $BE=q$, $CF=r$; then $r_1=\dfrac{S}{s_1}$, $r_2=\dfrac{S}{s_2}$, &c.,

and $\dfrac{1}{r_1}+\dfrac{1}{r_3}+\dfrac{1}{r_5}=\dfrac{s_1+s_3+s_5}{S}$

$$=\frac{\frac{1}{2}(a+\frac{1}{2}c+r)+\frac{1}{2}(c+\frac{1}{2}b+q)+\frac{1}{2}(b+\frac{1}{2}a+p)}{S}$$

$$=\frac{3(a+b+c)+2(p+q+r)}{4S}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}.$$

363. Let AC, BD, be the chords, intersecting in E;
$\angle BAC=\alpha$, $\angle ABD=\beta$:
then arc $CD=\pi-BC-AD=\pi-2\alpha-2\beta$;
and area DEC : area AEB

$$=\tfrac{1}{2}CD^2\frac{\sin C\sin D}{\sin(C+D)}:\tfrac{1}{2}AB^2\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

$=CD^2 : AB^2$ (since $\angle C=\beta$, $\angle D=\alpha$)

$=\text{chd}^2(\pi-2\alpha-2\beta) : 4r^2=\sin^2\{\tfrac{1}{2}\pi-(\alpha+\beta)\} : 1$
$=\cos^2(\alpha+\beta) : 1.$

64. $2\sin(45^\circ\pm x)=\sqrt{\{2-2\cos(90^\circ\pm 2x)\}}=\sqrt{(2\pm 2\sin 2x)}$;

$\therefore\ 2\sin 45^\circ(1-\tfrac{1}{2})=2\sin(45^\circ-\tfrac{1}{2}45^\circ)=\sqrt{(2-\sqrt{2})}$,

$$2\sin 45^\circ\left(1-\tfrac{1}{2}+\frac{1}{2^2}\right)=2\sin\{45^\circ-\tfrac{1}{2}45^\circ(1-\tfrac{1}{2})\}$$

$$=\sqrt{\{2-\sqrt{(2-\sqrt{2})}\}},\ \&\text{c.}$$

$$2\sin 45^\circ\left(1-\tfrac{1}{2}+\&\text{c.}\pm\frac{1}{2^n}\right)=\sqrt{\{2-\sqrt{(2-\&\text{c.})}\}}$$

$$=2\sin 45^\circ\frac{1\mp\frac{1}{2^n}}{\frac{3}{2}}=2\sin\left(30^\circ\mp\frac{30^\circ}{2^n}\right)=2\cos\left(60^\circ\pm\frac{30^\circ}{2^n}\right).$$

65. Here θ is the circular measure of n'', and $\theta=n\sin 1''$;

then $\cos\theta=1-\tfrac{1}{2}\theta^2$, or $\cos^{\frac{1}{3}}\theta=(1-\tfrac{1}{2}\theta^2)^{\frac{1}{3}}=1-\tfrac{1}{6}\theta^2$, nearly,

and $\theta\cos^{\frac{1}{3}}\theta=\theta-\tfrac{1}{6}\theta^3=\sin\theta$, nearly,

or (95) $n\sin 1''\cos^{\frac{1}{3}}\theta=\sin n''$;

$\therefore\ \log n=L\sin n''-L\sin 1''-\tfrac{1}{3}(L\cos\theta-10.)$

66. $a\sin\theta+b=c\cos\theta,\ a\cos\theta+b=d\sin\theta$;

$\therefore\ c\cos\theta-a\sin\theta=d\sin\theta-a\cos\theta$, or $\tan\theta=\dfrac{a+c}{a+d}$;

$\therefore$ since $a\tan\theta+b\sec\theta=c$,
we have $a(a+c)+b\sqrt{\{(a+c)^2+(a+d)^2\}}=c(a+d)$,
whence $b^2\{(a+c)^2+(a+d)^2\}=(a^2-cd)^2$

67. By (115 N. B.) distance from D of point of contact with AD of circle in $ABD=\tfrac{1}{2}(AD+DB-AB)=\tfrac{1}{2}(AD+s-b-c)$; and so likewise that of circle in

$$ACD=\tfrac{1}{2}(AD+DC-AC)=\tfrac{1}{2}(AD+s-c-b).$$

68. Observing throughout that
$\sqrt{2}\sin(45^\circ-A)=\sqrt{2}(\sin 45^\circ\cos A-\cos 45^\circ\sin A)=\cos A-\sin A$,
we have

$$u=2\sin\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)-(\cos\tfrac{1}{2}\psi-\sin\tfrac{1}{2}\psi)^2$$
$$=2\sin(45^\circ-\tfrac{1}{2}\psi)\cos\tfrac{1}{2}(\theta-\phi)-(\cos\tfrac{1}{2}\psi-\sin\tfrac{1}{2}\psi)^2$$
$$=(\cos\tfrac{1}{2}\psi-\sin\tfrac{1}{2}\psi)\{\sqrt{2}\cos\tfrac{1}{2}(\theta-\phi)-\sqrt{2}\sin(45^\circ-\tfrac{1}{2}\psi)\}$$
$$=\sqrt{2}(\cos\tfrac{1}{2}\psi-\sin\tfrac{1}{2}\psi)\{\cos\tfrac{1}{2}(\theta-\phi)-\cos\{90^\circ-\tfrac{1}{2}(\theta+\phi)\}\}$$
$$=\sqrt{2}(\cos\tfrac{1}{2}\psi-\sin\tfrac{1}{2}\psi)\{2\sin(45^\circ-\tfrac{1}{2}\theta)\sin(45^\circ-\tfrac{1}{2}\phi)\}=\&$$

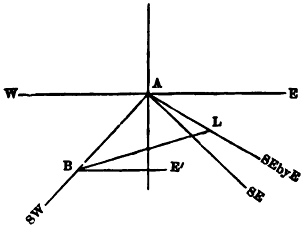
369. Let A, B, be the two points, and CD the given line, to which AB is inclined at an angle a, suppose:
let $AM=a$, $BN=b$, be the perpendiculars on CD;

then $\sin a=\dfrac{b-a}{c}$, and $mn=c\cos a$.

Produce AM to A', making $A'M=AM$, and join BA', cutting CD in E; then $AE+EB$ is less than the sum of any other pair of lines that can be drawn from A and B to any point in CD; (for if E' be any other such point, we have $AE'+E'B=A'E'+E'B>A'B$ or $AE+EB$.) Hence the perimeter of AEB is less than that of any other triangle on AB with its vertex in CD:
and area $AEB=\frac{1}{2}AE.EB\sin 2\theta$, (if $\angle AEM=\theta$,)
$=\frac{1}{2}(a\operatorname{cosec}\theta)(b\operatorname{cosec}\theta)\sin 2\theta=ab\cot\theta$:
but $mn=a\cot\theta+b\cot\theta=c\cos a$, as above;

$\therefore$ area $AEB=\dfrac{ab}{a+b}c\cos a=\dfrac{ab}{a+b}\sqrt{\{c^2-(a-b)^2\}}$

370.



A, B, the two positions of the ship, L the lighthouse; then
$\angle LAE=\frac{3}{4}\times 45°=33°45'$,
and drawing BE' due E from B,
$\angle LBE'=\frac{1}{4}\times 45°=11°15'$.
Hence, in triangle ABL, we have $AB=1\frac{1}{2}\times 8=12$,
$\angle ABL=\angle\angle(ABE'-LBE')=45°-11°15'=33°45'$
$\angle BAL=180°-BAW-LAE=101°15'$,
and $\angle ALB=180°-ABL-BAL=45°$;

and $AL=AB\dfrac{\sin 33°45'}{\sin 45°}=12\sqrt{2}\times .555=9.4$ *miles*,

$BL=AB\dfrac{\sin 101°15'}{\sin 45°}=11\sqrt{2}\times .980=15.2$ *miles*.

371. The circle through Pab will also pass through C, since a circle may be described round $PaCb$, and its centre will be C', *the middle* point of PC: so the centre of circle round Pac *will be* B', the middle point of PB:

hence $B'C' = \frac{1}{2}BC = \frac{1}{2}a$, and similarly of the other sides; so that the triangles ABC, $A'B'C'$, are similar, and their areas are as the squares of homologous sides or as $1 : \frac{1}{4}$. The proof holds whether P be external or internal.

372. $2\cos^{-1}\cot(2\tan^{-1}x) = n\pi$;
$\therefore \cot(2\tan^{-1}x) = \cos\frac{1}{2}n\pi = 0$ or ± 1;
and $2\tan^{-1}x = \frac{1}{2}m\pi$, or $m\pi \pm \frac{1}{4}\pi$;
$\therefore x = \tan\frac{1}{4}m\pi = 0, \infty$, or ± 1,
or $= \tan(4m \pm 1)\frac{1}{8}\pi = \pm(\sqrt{2} - 1)$ or $\pm(\sqrt{2} + 1)$.

373. $$u = \left(\frac{1}{ab} - \frac{1}{bc}\right)^2 - 2\left(\frac{1}{ab} - \frac{1}{bc}\right)\left(\frac{\cos ma}{ab} - \frac{\cos mc}{bc}\right) + \frac{\cos^2 ma}{a^2b^2}$$

$$+ \frac{\cos^2 mc}{b^2c^2} - 2\frac{\cos ma \cos mc}{ab^2c}$$

$$+ \frac{\sin^2 ma}{a^2b^2} + \frac{\sin^2 mc}{b^2c^2} - \frac{2\sin ma \sin mc}{ab^2c}$$

$$= \left(\frac{1}{ab} - \frac{1}{bc}\right)^2 - 2\left(\frac{1}{ab} - \frac{1}{bc}\right)\left(\frac{\cos ma}{ab} - \frac{\cos mc}{bc}\right)$$

$$+ \frac{1}{a^2b^2} + \frac{1}{b^2c^2} - \frac{2\cos(ma - mc)}{ab^2c}$$

$$= \frac{1}{abc}\left\{\frac{(c-a)^2}{abc} - 2\frac{c-a}{ab}\cos ma + 2\frac{c-a}{bc}\cos mc\right.$$

$$\left. + \frac{a^2 + c^2}{abc} - \frac{2\cos mb}{b}\right\}$$

$$= \frac{1}{abc}\left\{\frac{c^2 - 2ac + a^2}{abc} + 2\frac{\cos ma}{a} - 2\frac{\cos mc}{c} - 2\frac{\cos mb}{b}\right.$$

$$\left. + \frac{b^2 + 2ac}{abc}\right\}$$

$$= \frac{1}{abc}\left\{\frac{a^2 + b^2 + c^2}{abc} + 2\frac{\cos ma}{a} - 2\frac{\cos mc}{c} - 2\frac{\cos mb}{b}\right\}:$$

if $a = c$, and $b = 0$, $u = \dfrac{a^2 + b^2 + c^2}{a^2b^2c^2} - \dfrac{2}{ab^2c}\cos mb$

$$= \frac{1}{a^2c^2} - \frac{2}{ab^2c}\left(1 - \frac{m^2b^2}{2} + \&c.\right) = \frac{1}{a^4} + \frac{m^2}{a^2}.$$

374. Let AC, BD, intersect in E;
$\angle BAD=a$, $\angle BAC=\phi$, $\angle AED=\theta$;

then $\frac{AB}{AD}=\frac{a}{b}=\frac{\sin(\theta+a-\phi)}{\sin(\theta-\phi)}$ (1),

$\frac{AB}{BC}=\frac{a}{b}=\frac{\sin(a-\phi)}{\sin\phi}=\sin a\cot\phi-\cos a$;

$\therefore\ \tan\phi=\frac{b\sin a}{a+b\cos a}$;

and so from (1) $\tan(\theta-\phi)=\frac{b\sin a}{a-b\cos a}=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$;

$\therefore\ \tan\theta(a-b\cos a-b\sin a\tan\phi)=b\sin a+(a-b\cos a)\tan\phi$,

or $\tan\theta\left\{(a-b\cos a)-\frac{b^2\sin^2 a}{a+b\cos a}\right\}$

$=b\sin a+(a-b\cos a)\frac{b\sin a}{a+b\cos a}$, whence $\tan\theta$.

375. Let O_1 be the centre of the circle which touches BC in D, and the other two sides produced, o the centre of the circle inscribed within the exterior angle C, touching this circle in c, and BC in d:

then $od=r_c\text{(suppose)}=Co\cos\frac{1}{2}C=(Cc-r^c)\cos\frac{1}{2}C$
$=(OC-r_1-r_c)\cos\frac{1}{2}C$
$=\{r_1(\sec\frac{1}{2}C-1)-r_c\}\cos\frac{1}{2}C$;

or $r_c=r_1\frac{1-\cos\frac{1}{2}C}{1+\cos\frac{1}{2}C}=\frac{S}{s-a}\tan^2\frac{1}{4}C$; so $r_{c'}=\frac{S}{s-b}\tan^2\frac{1}{4}C$;

$r_b=\frac{S}{s-c}\tan^2\frac{1}{4}B$, $r_{b'}=\frac{S}{s-a}\tan^2\frac{1}{4}B$,

$r_a=\frac{S}{s-b}\tan^2\frac{1}{4}A$, $r_{a'}=\frac{S}{s-c}\tan^2\frac{1}{4}A$;

$\therefore\ r_ar_br_c=\frac{S^3}{(s-a)(s-b)(s-c)}\tan^2\frac{1}{4}A\tan^2\frac{1}{4}B\tan^2\frac{1}{4}C$

$=r_{a'}r_{b'}r_{c'}$.

376. $\frac{\tan\theta}{\tan\theta'}=\sqrt{\frac{\cos\beta-\cos a}{\cos a}}\div\sqrt{\frac{\cos\beta-\cos\alpha'}{\cos\alpha'}}=\frac{\tan\alpha}{\tan\alpha'}$, by question;

whence $\cos\beta\left(\dfrac{\cos\alpha}{\sin^2\alpha}-\dfrac{\cos\alpha'}{\sin^2\alpha'}\right)=\cot^2\alpha-\cot^2\alpha'$,

$$\text{or } \cos\beta=\frac{\cos^2\alpha-\cos^2\alpha'}{\cos\alpha(1-\cos^2\alpha')-\cos\alpha'(1-\cos^2\alpha)}$$

$$=\frac{\cos\alpha+\cos\alpha'}{1+\cos\alpha\cos\alpha'};$$

$$\therefore\ \tan^2\tfrac{1}{2}\beta\frac{1-\cos\beta}{1+\cos\beta}=\frac{(1-\cos\alpha)(1-\cos\alpha')}{(1+\cos\alpha)(1+\cos\alpha')}=\tan^2\tfrac{1}{2}\alpha\tan^2\tfrac{1}{2}\alpha'.$$

77. Let $\phi=\frac{1}{2}\tan^{-1}\dfrac{b}{a}$;

then $\dfrac{b}{a}=\tan 2\phi$, $\dfrac{a}{b}=\tan\left(\dfrac{\pi}{2}-2\phi\right)$, $\frac{1}{2}\tan^{-1}\dfrac{a}{b}=\dfrac{\pi}{4}-\phi$:

$$\therefore\ u=\tfrac{1}{2}a^3\operatorname{cosec}^2(\tfrac{1}{4}\pi-\phi)+\tfrac{1}{2}b^3\sec^2\phi$$

$$=\frac{a^3}{1-\cos(\frac{1}{2}\pi-2\phi)}+\frac{b^3}{1+\cos 2\phi}=\frac{a^3}{1-\sin 2\phi}+\frac{b^3}{1+\cos 2\phi}$$

$$=\frac{a^3\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}-b}+\frac{b^3\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}+a}$$

$$=a\sqrt{(a^2+b^2)}\{\sqrt{(a^2+b^2)}+b\}$$
$$+b\sqrt{(a^2+b^2)}\{\sqrt{(a^2+b^2)}-a\}$$
$$=(a+b)(a^2+b^2).$$

78. Let A, B, be the initial positions of the ships, A', B', their positions at the end of time t, the lines AB, $A'B'$ being produced to meet in C at an angle θ:
then $CA=a$, $CB=b$, $CA'=a+ut$, $CB'=b+vt$,
and $A'B'^2=(a+ut)^2+(b+vt)^2-2(a+ut)(b+vt)\cos\theta$
$=(a^2+b^2-2ab\cos\theta)+2\{au+bv+(av+bu)\cos\theta\}t$
$+(u^2+v^2-2uv\cos\theta)t^2$
$=A+2Bt+Ct^2$ suppose $=$ (a *minimum*) m^2, suppose:
$\therefore\ Ct=-B\pm\sqrt{\{B^2-(A-m^2)C\}}$, which is impossible,

if $AC-B^2>m^2C$, or $m<\sqrt{\dfrac{AC-B^2}{C}}$, which is, therefore,

the least value of $m=$ (by putting the values of A, B, C, and

simplifying) $\dfrac{(av\backsim bu)\sin\theta}{\sqrt{(u^2+v^2-2uv\cos\theta)}}$.

379. $\tan\theta - \tan\theta' = \dfrac{1}{\cot\alpha + 2\sec 2\phi} - \dfrac{1}{3\sin\alpha' + \sin^2\phi}$

$$= \frac{\cos 2\phi}{\cot\alpha\cos 2\phi + 2} - \frac{2}{6\sin\alpha' + 1 - \cos 2\phi}$$

$=$ (a *constant*) a, suppose, for all values of ϕ;

$\therefore\ (6\sin\alpha' + 1)\cos 2\phi - \cos^2 2\phi - 2\cot\alpha\cos 2\phi - 4$

$= 2a(6\sin\alpha' + 1) + a\cos 2\phi\{(6\sin\alpha' + 1)\cot\alpha - 2\}$

$- a\cot\alpha\cos^2 2\phi$:

hence, equating coefficients of the different powers of the variable $\cos 2\phi$, we get

(i) $-1 = -a\cot\alpha$, whence $a = \tan\alpha$,

(ii) $6\sin\alpha' + 1 - 2\cot\alpha = a(6\sin\alpha' + 1)\cot\alpha - 2a$,

(iii) $-4 = 2a(6\sin\alpha' + 1)$, whence (putting for a its value, $\tan\alpha$) we have $6\sin\alpha' + 1 = -2\cot\alpha$, and now (ii) gives

$$-2\cot\alpha - 2\cot\alpha = \tan\alpha(-2\cot\alpha)\cot\alpha - 2\tan\alpha,$$

whence $\tan\alpha = \pm 1$, and $\therefore \sin\alpha' = -\frac{1}{2}$ or $\frac{1}{6}$:

hence $\tan\theta - \tan\theta' = \tan\alpha = \pm 1$, for all values of ϕ,

if $\alpha = \pm 45°$, $\alpha' = -30°$ or $+\sin^{-1}(\frac{1}{6})$.

380. Let r be the radius of the circle;

then (115) $rs' = S' = \frac{1}{2}a'b'\sin C'$, and (114) $a = 2r\sin A$;

$$\therefore\ \frac{a}{a'} = \frac{b'}{s'}\sin A\sin C' = (110)\ \frac{2\sin B}{\sin A + \sin B + \sin C}\sin A\sin C$$

(since $A' = A$, &c., the triangles being similar)

$= $ (see Ex. 15. 4) $4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$.

381. Let x be the height of the balloon P; then $AP = x\cot\alpha$, $BP = x\cot\beta$, $CP = x\cot\gamma$; but $\cos ABP = -\cos CPB$,

or $\dfrac{a^2 + x^2\cot^2\beta - x^2\cot^2\alpha}{2ax\cot\beta} = -\dfrac{b^2 + x^2\cot^2\beta - x^2\cot^2\gamma}{2bx\cot\beta}$,

whence x^2.

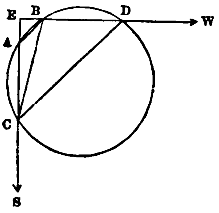
382. $\cos^{-1}\left(1 - \dfrac{bx}{a}\right) = \cos^{-1}\left(1 - \dfrac{x}{a}\right) - \cos^{-1} b = \theta - \phi$, suppose;

$$\therefore\ 1 - \frac{bx}{a} = \cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$= \left(1 - \frac{x}{a}\right)b + \frac{\sqrt{(2ax - x^2)}}{a}\sqrt{(1 - b^2)};$$

$$\therefore\ a\frac{1 - b}{\sqrt{(1 - b^2)}} = a\sqrt{\frac{1 - b}{1 + b}} = \sqrt{(2ax - x^2)},$$

whence $x^2-2ax+a^2=a^2\left(1-\frac{1-b}{1+b}\right)=a^2\frac{2b}{1+b}$, &c.

83. 

AB the wall, C, D, the two places of observation;

then $\angle ACB=a=\angle ADB$, and C, D, are points in the circumference of the circle, circumscribing either of the triangles ABC, ABD; also AS and BW produced backwards must meet at right angles in E;

$\therefore \angle CBD=\angle\angle(BEC+BCE)=90°+a$;

hence diam. of circle (114) $=\frac{AB}{\sin a}=\frac{CD}{\cos a}$, or $AB=CD\tan a$.

84. $\cos a\cos x+\sin a\sin x=\cos a\cos b$:

$\therefore$ 1st Approximation, $\cos a+x\sin a=\cos a\cos b$,

or $x=\cot a(\cos b-1)=-2\cot a\sin^2\frac{1}{2}b$;

$\therefore$ 2nd App. $\cos a(1-\frac{1}{2}x^2)+x\sin a=\cos a\cos b$;

$\therefore x\sin a=\cos a\cos b-\cos a(1-\frac{1}{2}x^2)$

$$\begin{aligned}\text{or } x&=-2\cot a\sin^2\tfrac{1}{2}b+\tfrac{1}{2}x^2\cot a\\&=-2\cot a\sin^2\tfrac{1}{2}b\,(1-\tfrac{1}{2}x\cot a)\\&=-2\cot a\sin^2\tfrac{1}{2}b\,\{1-\cot^2 a\sin^2\tfrac{1}{2}b\}\text{, by 1st App.}\end{aligned}$$

To obtain x in *seconds*, multiply this result by $\frac{180°}{\pi}$,

or (95) divide it by $\sin 1''$.

85. $m^2=\frac{1}{1+3\cos^2\phi}=\frac{1}{1+3\left(\frac{1-t^2}{1+t^2}\right)^2}$, by (76), putting t for $\tan\frac{1}{2}\phi$,

$$=\frac{(1+t^2)^2}{4(1-t^2+t^4)}=\frac{(1+t^2)^3}{4(1+t^6)}=\frac{(1+\tan^{\frac{2}{3}}a)^3}{4(1+\tan^2 a)}=\tfrac{1}{4}(\cos^{\frac{2}{3}}a+\sin^{\frac{2}{3}}a)^3.$$

86. Since $\sin A=\sin\{n\pi+(-1)^n A\}$, the value of $\tan\frac{1}{4}A$ in terms of $\sin A$ must include all the values of $\tan\frac{1}{4}\{n\pi+(-1)^n A\}$

$=$ (i) $\tan(\frac{1}{2}m\pi+\frac{1}{4}A)$, (if n be *even* and $=2m$,)

$=+\tan\frac{1}{4}A$ or $-\cot\frac{1}{4}A$, according as m is *even* or *odd*,

$=$ (ii) $\tan(\frac{1}{2}m\pi+\frac{1}{4}(\pi-A))$, (if n be odd and $=2m+1$,)

$=+\tan\frac{1}{4}(\pi-A)$, or $-\cot\frac{1}{4}(\pi-A)=-\tan\frac{1}{4}(\pi+A)$, according as m is *even* or *odd*.

387. Draw AP, BQ, CR, perp. on BC, AC, AB, intersecting in D: then P will be the point of intersection of circles on AB, AC, since each of the angles APB, APC, is a right angle; and similarly for Q and R.

Now $BC =$ diam. of circle on $BC = \dfrac{QR}{\sin QBR} = \dfrac{QR}{\cos A}$, &c.

$\therefore\ QR + PR + PQ = a\cos A + b\cos B + c\cos C$:

$$\therefore\ \frac{\text{perimeter of } ABC}{\text{perimeter of } PQR} = \frac{a+b+c}{a\cos A + b\cos B + c\cos C}$$

$$= (110 \text{ Ex. } \mathbf{1.}) \frac{\sin A + \sin B + \sin C}{\sin A\cos A + \sin B\cos B + \sin C\cos C}$$

$$= \frac{2(\sin A + \sin B + \sin C)}{\sin 2A + \sin 2B + \sin 2C}$$

$$= \frac{2(\sin A + \sin B + \sin C)}{2\sin A\cos A + 2\sin(B+C)\cos(B-C)}$$

$$= \frac{\sin A + \sin B + \sin C}{\sin A\,\{\cos(B-C) - \cos(B+C)\}}$$

$$= (\text{Ex. } \mathbf{15.}\ 4) \frac{4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin A\,(2\sin B\sin C)} = \&\text{c.}$$

388. $\cos^2 3x = 1 - \sin 2x = (\cos x \backsim \sin x)^2$;

$\therefore$ (i) $\sin x = \cos x - \cos 3x = 2\sin x \sin 2x$,
whence $\sin x = 0$ or $x = n\pi$, and $\sin 2x = \frac{1}{2}$,
or $2x = 30°$ or $n\pi + (-1)^n \frac{1}{6}\pi$;

(ii) $\sin x = \cos x + \cos 3x = 2\cos x\cos 2x$,

$$\text{or } \tan x = 2\cos 2x = 2\frac{1-\tan^2 x}{1+\tan^2 x},$$

whence $\tan^3 x + 2\tan^2 x + \tan x = 2$, a cubic equation.

389. Let $A = A' + \alpha$, $B = B' + \beta$, where A', B', are the *true* angles of the triangle, and α, β, are small errors made in observing;

then $\sin A = \sin A'\cos\alpha + \cos A'\sin\alpha = \sin A' + \alpha\cos A'$, nearly;
$\cos A = \cos A'\cos\alpha - \sin A'\sin\alpha = \cos A' - \alpha\sin A'$; &c.
and $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= (\sin A' + \alpha\cos A')(\cos B' - \beta\sin B')$$
$$+ (\cos A' - \alpha\sin A')(\sin B' + \beta\cos B')$$

$= \sin(A'+B') + \alpha\cos(A'+B') + \beta\cos(A'+B') +$ terms in $\alpha\beta$, which are small, in comparison with terms in (α) or (β): hence the error is least, that is $(A+B)$ most nearly $=(A'+B')$, when $\cos(A'+B')$ is near 0, that is, when $(A'+B')$ or $(A+B)$ is near 90°, and $\therefore\ C=90^\circ$, nearly.

90. Here $\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma$

$$= \frac{\sin(\alpha-\theta)}{\sin\alpha\sin\theta} \times \frac{\sin(\beta-\theta)}{\sin\beta\sin\theta} \times \frac{\sin(\gamma-\theta)}{\sin\gamma\sin\theta}$$

$$= (\cot\theta - \cot\alpha)(\cot\theta - \cot\beta)(\cot\theta - \cot\gamma)$$
$$= \cot^3\theta - (\cot\alpha + \cot\beta + \cot\gamma)\cot^2\theta$$
$$+ (\cot\alpha\cot\beta + \cot\alpha\cot\gamma + \cot\beta\cot\gamma)\cot\theta - \cot\alpha\cot\beta\cot\gamma:$$

but (Ex. **15.** 2) $\cot\alpha\cot\beta + \&c. = 1$;

and, since $\cos(\alpha+\beta+\gamma) = -1$

$$= \cos(\alpha+\beta)\cos\gamma - \sin(\alpha+\beta)\sin\gamma$$
$$= \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\cos\gamma - \sin\alpha\sin\gamma\cos\beta$$
$$- \sin\beta\sin\gamma\cos\alpha$$

$\therefore\ \cot\alpha\cot\beta\cot\gamma + \operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma = \cot\alpha + \cot\beta + \cot\gamma$;

hence we get

$$\cot^2\theta\{\cot\theta - (\cot\alpha + \cot\beta + \cot\gamma)\}$$
$$+ \cot\theta - (\cot\alpha + \cot\beta + \cot\gamma) = 0;$$

$\therefore\ \cot\theta = \cot\alpha + \cot\beta + \cot\gamma$,

(the other solution, $\cot^2\theta + 1 = 0$, being impossible.)

Also $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + (\cot\alpha + \cot\beta + \cot\gamma)^2$

$= 3 + \cot^2\alpha + \cot^2\beta + \cot^2\gamma$, by (Ex. **15.** 2)

$= \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma$.

91. $2R = \dfrac{a}{\sin A} = \dfrac{a'}{\sin A'}$; but $A' = \angle$ at circumference on

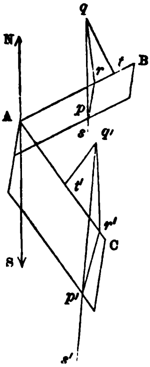
(arc $B'AC'=$) $\frac{1}{2}AC + \frac{1}{2}AB = \frac{1}{2}B + \frac{1}{2}C = 90^\circ - \frac{1}{2}A$;

$$\therefore\ a' = a\frac{\sin A'}{\sin A} = \frac{a}{2\sin\frac{1}{2}A},$$

and area $A'B'C' = \frac{1}{2}a'b'\sin C' = \frac{1}{2}\cdot\dfrac{a}{2\sin\frac{1}{2}A}\cdot\dfrac{b}{2\sin\frac{1}{2}B}\cos\frac{1}{2}C$

$$= \frac{\text{area } ABC}{8\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}.$$

392. AB, AC the bases of the two walls (which, however, need not, as in the figure, be produced so as to *join* at A;) sq, $s'q'$, the courses of two rays of the sun at *noon*, when the sun is due south, passing in parallel lines over the walls at p, p', where the vertical lines $pr, p'r'$, in the walls will cast shadows $rq, r'q'$, towards the north: then qt, $q't'$, perp. to AB, AC, will be the *widths* of the shadows of the two walls.



And $qt = qr\sin\beta = pr\cot\alpha\sin\beta$,

$$\text{or } b = a\cot\alpha\sin\beta;$$

$$q't' = q'r'\cos\beta = p'r'\cot\alpha\cos\beta,$$

$$\text{or } b' = a'\cot\alpha\cos\beta;$$

$$\therefore \frac{b^2}{a^2} + \frac{b'^2}{a'^2} = \cot^2\alpha(\cos^2\beta + \sin^2\beta) = \cot^2\alpha, \text{ and } \cot\beta = \frac{ab'}{a'b}.$$

393. O, D, E, F, the centres of the inscribed and escribed circles Then, since $OBD = 90° = OCD$, a circle may be described round $OBDC$, and similarly round $OAFB$, $OCEA$.

$$\text{Now diam. of circle } ODE = \frac{OD}{\sin OED} = \frac{\text{diam. of } OBDC}{\sin OAC}$$

(since $\angle\angle\, OAC$, OEC, are in the same segment)

$$= \frac{BC}{\sin BOC} \times \frac{1}{\sin OAC} = \frac{a}{\cos\frac{1}{2}A} \times \frac{1}{\sin\frac{1}{2}A}$$

$$(\text{since } BOC = 180° - \tfrac{1}{2}B - \tfrac{1}{2}C = 90° + \tfrac{1}{2}A) = \frac{2a}{\sin A} = 2R;$$

and similarly for diam. of $OAEC$ and $OAFB$.

$$\text{Also diam. of } DEF = \frac{DE}{\sin F} = \frac{DE}{\sin DOE}$$

$$(\text{since } F = 180° - AOB = 180° - DOE)$$

$$= \text{diam. of } ODE = 2R, \text{ as before.}$$

394. (i) If Q be *within* the circle, we have

$$\angle QAB = \tfrac{1}{2}A,\ \angle PAB = 90° - C,\ \angle PAQ = (90° - C) \sim \tfrac{1}{2}A;$$

$$PQ^2 = AP^2 + AQ^2 - 2AP.AQ\cos PAQ$$

$$= R^2 + r^2\operatorname{cosec}^2\tfrac{1}{2}A - 2Rr\operatorname{cosec}\tfrac{1}{2}A\cos(90° - C - \tfrac{1}{2}A)$$

$$=R^2+4Rr\operatorname{cosec}\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$
$$-2Rr\operatorname{cosec}\tfrac{1}{2}A\cos\tfrac{1}{2}(B\backsim C),\text{ as in (Ex. 20.8)}$$
$$=R^2-2Rr\operatorname{cosec}\tfrac{1}{2}A\{\cos\tfrac{1}{2}(B-C)-2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\}$$
$$=R^2-2Rr\operatorname{cosec}\tfrac{1}{2}A\cos\tfrac{1}{2}(B+C)=R^2-2Rr.$$

(ii) If Q be *without* the circle, we have

$$\angle QAB=90^\circ-\tfrac{1}{2}A,\ \angle PAQ=180^\circ-C-\tfrac{1}{2}A$$
$$=90^\circ+\tfrac{1}{2}(B-C);$$
$$PQ^2=R^2+r'^2\sec^2\tfrac{1}{2}A-2Rr'\sec\tfrac{1}{2}A\cos\{90^\circ+\tfrac{1}{2}(B-C)\}$$
$$=R^2+2Rr'\sec\tfrac{1}{2}A\{2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}B+\sin\tfrac{1}{2}(B-C)\}$$
$$=R^2+2Rr'\sec\tfrac{1}{2}A\sin\tfrac{1}{2}(B+C)=R^2+2Rr'.$$

95. $PQ^2+PQ_1{}^2+PQ_2{}^2+PQ_3{}^2$

$$=4R^2+2R(r_1+r_2+r_3-r)=12R^2,\text{ by (Ex. 327.)}$$

96. $\sin\theta=2\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta=2^2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{4}\theta\sin\tfrac{1}{4}\theta=\&c.$

$$=2^n\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\ \ldots\ \cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}$$

$$=\theta\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ \ldots\ \cos\frac{\theta}{2^n}\times\left(\sin\frac{\theta}{2^n}\div\frac{\theta}{2^n}\right);$$

but as n increases, and $\frac{\theta}{2^n}$ diminishes,

$$\sin\frac{\theta}{2^n}\div\frac{\theta}{2}\text{ tends to 1, by (91);}$$

$\therefore\ \sin\theta=\theta\cos\tfrac{1}{2}\theta\cos\tfrac{1}{4}\theta\cos\tfrac{1}{8}\theta\ \ldots$ *ad infinitum.*

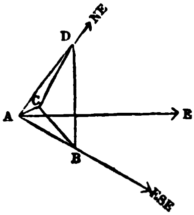
97. Let $\theta=\text{vers}^{-1}\frac{x}{a}$; $\therefore\ \cos\theta=1-\frac{x}{a}$, and $\tan\theta=\frac{\sqrt{(2ax-x^2)}}{a-x}$,

and [8] $\frac{1}{\sqrt{x}}\theta=\frac{1}{\sqrt{x}}(\tan\theta-\tfrac{1}{3}\tan^3\theta+\&c.)$

$$=\frac{1}{\sqrt{x}}\left\{\frac{\sqrt{(2ax-x^2)}}{a-x}-\&c.\right\}=\frac{\sqrt{(2a-x)}}{a-x}-\&c.=\sqrt{\frac{2}{a}},$$

when $x=0$.

98.



A the spectator, BC the column, $\angle EAB=22\tfrac{1}{2}^\circ$, $\angle CAB=a$, BD the shadow, which lies due N, since at noon the sun is due S; D the extremity of the shadow in the N.E, so that $\angle DAE=45^\circ$: then $AD=a$, $\angle ADB=45^\circ$, $\angle DAB=67\tfrac{1}{2}^\circ$, $\angle DBA=180^\circ-45^\circ-67\tfrac{1}{2}^\circ=67\tfrac{1}{2}^\circ$

$$=\angle DAB;$$

$\therefore AB = BC \cot\alpha = (\text{also})\, 2AD\cos DAB = 2a\sin 22\tfrac{1}{2}^\circ$
$= a\sqrt{(2 - 2\cos 45^\circ)} = a\sqrt{(2 - \sqrt{2})}$, whence BC.

399. $\dfrac{1+\cos\theta}{\cos\theta} \cdot \dfrac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} \ldots = \dfrac{2\cos^2\frac{1}{2}\theta}{\cos\theta} \cdot \dfrac{2\cos^2\frac{1}{4}\theta}{\cos\frac{1}{2}\theta} \ldots$

$= \dfrac{2^n}{\cos\theta}\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}$; $\therefore$ $\cos\dfrac{\theta}{2^n} = \tan\theta + \sin\dfrac{\theta}{2^n}$, as in (Ex. 396).

400. Here $a + c = 2b$, and $d^2 = ac =$ by (Ex. 394) $R^2 - 2Rr$

$= \dfrac{a^2b^2c^2}{16S^2} - \dfrac{abc}{2s} =$ by (114), where $s = \frac{1}{2}(a+b+c) = \frac{3}{2}b$,

and $S = \sqrt{\{\frac{3}{2}b(\frac{3}{2}b - a)\frac{1}{2}b(\frac{3}{2}b - c)\}}$
$= \frac{1}{4}b\sqrt{3\{9b^2 - 6b(a+c) + 4ac\}}$
$= \frac{1}{4}b\sqrt{3(4ac - 3b^2)}$:

putting these values for S and s in the above, we get $4b^2 = 5ac$;
hence $(a+c)^2 = 5ac$, or $a^2 - 3ac + c^2 = 0$,

whence $\dfrac{a}{c} = \dfrac{3-\sqrt{5}}{2} = \dfrac{\sqrt{5}-1}{\sqrt{5}+1}$, and $\dfrac{a}{a+c} = \dfrac{a}{2b} = \dfrac{\sqrt{5}-1}{2\sqrt{5}}$,

or $a : b : c :: \sqrt{5} - 1 : \sqrt{5} : \sqrt{5} + 1$.

401. Let A', B', C', be the vertices of the triangles on BC, AC, AB, respectively, D, E, F, the centres of their circumscribing circles; then $AE =$ rad. of circle, centre E,

$= \dfrac{BC}{\sin A'} = \dfrac{BC}{\sin A} =$ rad. of circle circumscribing ABC, and so

for the other circles: hence $AE = AF$,
and $\angle EAC = 90^\circ - B$, $\angle FAB = 90^\circ - C$,
$\angle EAF = A + (90^\circ - B) + (90^\circ - C) = 2A$,

and $EF = 2AE\sin\frac{1}{2}EAF = \dfrac{a}{\sin A}\sin A = a$:

so $DF = b$, $DE = c$.

402. Here $\sin(\theta + \alpha) + \cos\alpha\cos\theta\sin(\theta+\alpha) + \cos\alpha\sin\alpha\cos(\theta+\alpha)$
$+ \sin\alpha\cos\theta\cos(\theta+\alpha) = 0$;
$\therefore 2\sin(\theta+\alpha) + 2\cos\theta\{\sin(\theta+\alpha)\cos\alpha + \cos(\theta+\alpha)\sin\alpha\}$
$+ \sin 2\alpha\cos(\theta+\alpha) = 0$,
or $2\sin(\theta+\alpha) + 2\cos\theta\sin(\theta+2\alpha) + \sin 2\alpha\cos(\theta+\alpha) = 0$,
or $2\sin(\theta+\alpha) + \{\sin 2(\theta+\alpha) + \sin 2\alpha\} + \sin 2\alpha\cos(\theta+\alpha) = 0$:
$\therefore 2\sin(\theta+\alpha) + \{1 + \cos(\theta+\alpha)\}$
$+ \sin 2\alpha\{1 + \cos(\theta+\alpha)\} = 0$,

whence (i) $\cos(\theta+a)=-1=\cos(2n+1)\pi$,

or (ii) $2\sin(\theta+a)=-\sin 2a=-2\sin a\cos a$;

or $\theta=-a+\sin^{-1}(-\sin a\cos a)$.

03. If $a>45^\circ$, u would be impossible; but, if $a<45^\circ$,

then $\left(\text{since } \cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}, \text{ or } 2A=\cos^{-1}\dfrac{1-\tan^2 A}{1+\tan^2 A}\right)$

we have $u=2\tan^{-1}x$ (suppose) $=\cos^{-1}\dfrac{1-x^2}{1+x^2}$

$$=\cos^{-1}\frac{1-\tan(45^\circ-a)\tan^2\frac{1}{2}\beta}{1+\tan(45^\circ-a)\tan^2\frac{1}{2}\beta}$$

$$=\cos^{-1}\frac{(1+\tan a)\cos^2\frac{1}{2}\beta-(1-\tan a)\sin^2\frac{1}{2}\beta}{(1+\tan a)\cos^2\frac{1}{2}\beta+(1-\tan a)\sin^2\frac{1}{2}\beta}=\&\text{c.}$$

04. Let θ be the elevation of the tower;

then $h=300\tan\theta$, $h+8=300\tan(\theta+a)$, (if a be the circular measure of $57'17''.75$ or $3437''.75=\dfrac{3437.75}{206265}=\frac{1}{60}$),

$$=300\frac{\tan\theta+a}{1-a\tan\theta}=300\frac{h+300a}{1-ah}=h\left(1+\frac{300}{h}a\right)\left(1+\frac{h}{300}a\right)$$

$$=h+300a+\frac{h^2}{300}a, \text{ omitting terms in } a^2, \&\text{c.}:$$

hence $h^2=300\left(\dfrac{8}{a}-300\right)=300\times 180=54000$;

hence $h=\sqrt{54000}=232$ *feet*, nearly.

05. $AD=x$, $BE=y$, $CF=z$;

then (1) $b^2=x^2+CD^2-2x.CD\cos D$,

and (2) $c^2=x^2+BD^2+2x.BD\cos D$;

also (Euc. vi. 3) $\dfrac{BD}{CD}=\dfrac{c}{b}$, or $\dfrac{BD}{BC}=\dfrac{c}{b+c}$, $BD=\dfrac{ac}{b+c}$, $CD=\dfrac{ab}{b+c}$;

hence, multiplying (1) by BD and (2) by CD, and adding,

$$b^2\frac{ac}{b+c}+c^2\frac{ab}{b+c}=x^2a+a\frac{a^2bc}{(b+c)^2},$$

whence $(b+c)^2\dfrac{x^2}{bc}=(b+c)^2-a^2=(a+b+c)(b+c-a)$;

so $(a+c)^2\dfrac{y^2}{ac}=(a+c)^2-b^2=(a+b+c)(a+c-b)$, and $(a+b)^2\dfrac{z^2}{ab}=(a+b)^2-c^2=(a+b+c)(a+b-c)$;

the sum of which is $(a+b+c)^2$.

405. The radii of the circles, whose centres are A, B, C,
being $\frac{1}{2}(\sqrt{3}-1)$, $\frac{1}{2}(\sqrt{3}+1)$, $\frac{1}{2}(3-\sqrt{3})$,
we have $AB=\sqrt{3}$, $BC=2$, $AC=1$;

$$\therefore \cos A=\frac{3+1-4}{2\sqrt{3}}=0, \text{ or } A=90^\circ:$$

and $\sin B=\frac{1}{2}$, or $B=30^\circ$, $C=60^\circ$:
hence sector $A=$ (by (31) $\frac{1}{2}\theta r^2=$) $\frac{1}{8}\pi(2-\sqrt{3})$,
sector $B=(2+\sqrt{3})\frac{1}{24}\pi$, sector $C=(2-\sqrt{3})\frac{1}{4}\pi$;
and triangle $ABC=\frac{1}{2}AB.AC=\frac{1}{2}\sqrt{3}$:
$\therefore$ area between circles
$=\frac{1}{2}\sqrt{3}-\frac{1}{24}\pi\{(6-3\sqrt{3})+(2+\sqrt{3})+(12-6\sqrt{3})\}$
$=\frac{1}{2}\sqrt{3}-(5-2\sqrt{3})\frac{1}{6}\pi=\frac{43}{100}$ *sq. yds.*

406. $ABCD$ the base, O the vertex, $AB=a=OA$; OE perp. on AB, and BF, DF, perp. on OA: then, if OG be drawn to the centre of base, $\cos\theta=\dfrac{EG}{OE}=\dfrac{\frac{1}{2}a}{a\sin 60^\circ}=\dfrac{1}{\sqrt{3}}$,

and $\cos\phi=\cos BFD=\dfrac{BF^2+DF^2-BD^2}{2BF.DF}$:

now $BF=DF=a\sin 60^\circ=\frac{1}{2}a\sqrt{3}$,
and $BD^2=AB^2+AD^2=2a^2$;

$$\therefore \cos\phi=\frac{\frac{3}{2}a^2-2a^2}{\frac{3}{2}a^2}=1-\tfrac{4}{3}=-\tfrac{1}{3}=-\cos^2\theta.$$

407. We have here to expel γ from the first result:

$$\text{hence } u=\frac{\cos\alpha+\cos(\alpha+\beta)-\sin\beta}{\cos\beta+\cos(\alpha+\beta)-\sin\alpha}$$

$$=\frac{2\cos(\alpha+\frac{1}{2}\beta)\cos\frac{1}{2}\beta-\sin\beta}{2\cos(\beta+\frac{1}{2}\alpha)\cos\frac{1}{2}\alpha-\sin\alpha}$$

$$=\frac{\cos\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha}\cdot\frac{\cos(\alpha+\frac{1}{2}\beta)-\cos(90^\circ-\frac{1}{2}\beta)}{\cos(\beta+\frac{1}{2}\alpha)-\cos(90^\circ-\frac{1}{2}\alpha)}$$

$$=\frac{\cos\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha}\cdot\frac{2\sin(45^\circ+\frac{1}{2}\alpha)\sin\{45^\circ-\frac{1}{2}(\alpha+\beta)\}}{2\sin(45^\circ+\frac{1}{2}\beta)\sin\{45^\circ-\frac{1}{2}(\alpha+\beta)\}}$$

$$=\frac{\cos\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha}\cdot\frac{\cos\frac{1}{2}\alpha+\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\beta+\sin\frac{1}{2}\beta}=\&\text{c.}$$

408. $OQ_1=OP_1\cos 60^\circ=\frac{1}{2}r$; assume $OQ_{n-1}=\frac{1}{n}r$;

let $OP_{n+2}Q_n=\theta$: then $\dfrac{OP_{n+2}}{OQ_{n-1}}=\dfrac{r}{OQ_{n-1}}=\dfrac{\sin(60^\circ-\theta)}{\sin\theta}$,

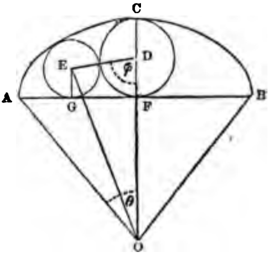
$$\frac{OP_{n+2}}{OQ_n}=\frac{r}{OQ_n}=\frac{\sin(60^\circ+\theta)}{\sin\theta};$$

$$\therefore \frac{r}{OQ_n}-\frac{r}{OQ_{n-1}}=2\cos 60^\circ=1,$$

and $\dfrac{r}{OQ_n}=\dfrac{r}{OQ_{n-1}}+1=n+1$, or $OQ_n=\dfrac{r}{n+1}$.

Hence, since $OQ_1=\frac{1}{2}r$, we have $OQ_2=\frac{1}{3}r$, $OQ_3=\frac{1}{4}r$, &c.

10.



$$DF=EG+ED\cos\phi,$$

or $r'=r''+(r''+r')\cos\phi$;

$$\therefore \cos\phi=\frac{r'-r''}{r'+r''}$$

$$=(\text{also})\,\frac{OD^2+DE^2-OE^2}{2OD.DE}$$

$$=\frac{(r-r')^2+(r'+r'')^2-(r-r'')^2}{2(r-r')(r'+r'')}:$$

$$\therefore 2(r-r')(r'-r'')$$

$$=2r'^2-2rr'+2r'r''+2rr'',$$

whence (multiplying out) we get $r''=r'-\dfrac{r'^2}{r}=r'-r'\sin^2\frac{1}{2}\theta$,

(since $2r'=r\,\text{vers}\,\theta=r\sin^2\frac{1}{2}\theta$,) $=r'\cos^2\frac{1}{2}\theta=r\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta$

$=\frac{1}{4}r\sin^2\dfrac{l}{2r}$, since $\theta=\dfrac{l}{2r}$.

11. Put $\theta=\sin^{-1}(3\sin\alpha)$;
then we have to show that $\tan^3\frac{1}{4}(\theta+\alpha)=\tan\frac{1}{4}(\theta-3\alpha)$:
now $\sin\theta-\sin\alpha=2\sin\alpha=(\text{also})\,2\sin\frac{1}{2}(\theta-\alpha)\cos\frac{1}{2}(\theta+\alpha)$;

hence $\tan^2\frac{1}{4}(\theta+\alpha)=$ by (76) $\dfrac{1-\cos\frac{1}{2}(\theta+\alpha)}{1+\cos\frac{1}{2}(\theta+\alpha)}$

$$=\frac{\sin\frac{1}{2}(\theta-\alpha)-\sin\alpha}{\sin\frac{1}{2}(\theta-\alpha)+\sin\alpha}=\frac{\tan\frac{1}{4}(\theta-3\alpha)}{\tan\frac{1}{4}(\theta+\alpha)}.$$

12. Put $2\cos\theta=x+\dfrac{1}{x}$; then $x^7+\dfrac{1}{x^7}+7\left(x+\dfrac{1}{x}\right)=0$,

whence $x+\dfrac{1}{x}=0$, which gives $\cos\theta=0$, or $\theta=(2n+1)\frac{1}{2}\pi$;

and also $x^6 - x^4 + x^2 - 1 + \frac{1}{x^2} - \frac{1}{x} + \frac{1}{x^6} + 7 = 0$,

which gives $2\cos 6\theta - 2\cos 4\theta + 2\cos 2\theta + 6 = 0$,
or $\cos 6\theta =$ by (75) $4\cos^3 2\theta - 3\cos 2\theta = \cos 4\theta - \cos 2\theta - 3$
$= 2\cos^2 2\theta - \cos 2\theta - 4$:
$\therefore\ 2(1 + \cos^3 2\theta) = \cos 2\theta (1 + \cos 2\theta)$, or $\cos 2\theta = -1$,
whence $2\theta = (2n + 1)\pi$, as before;
and $2(1 - \cos 2\theta + \cos^2 2\theta) = \cos 2\theta$, where $\cos 2\theta$ is impossible.

413. $u = 2\cos \tfrac{1}{2}n(A + B)\cos \tfrac{1}{2}n(A - B) + \cos nC$:

(i) If n be even, and $= 2m$,

then $\cos \tfrac{1}{2}n(A + B) = \cos m(A + B) = \cos(m\pi - mC)$
$= -(-1)^m \cos mC$,
and $\cos nC = \cos 2mC = 2\cos^2 mC - 1$;
$\therefore\ u = -1 + 2\cos mC\{\cos mC - (-1)^m \cos m(A - B)\}$
$= -1 - 2(-1)^m \cos mC\{\cos m(A + B)$
$+ \cos m(A - B)\}$
$= -1 \pm 4\cos mC \cos mA \cos mB$
$= -1 \pm 4\cos \tfrac{1}{2}nA \cos \tfrac{1}{2}nB \cos \tfrac{1}{2}nC$.

(ii) If n be odd, and $= 2m + 1$,

then $\cos \tfrac{1}{2}n(A + B) = \cos\{m\pi + (\tfrac{1}{2}\pi - \tfrac{1}{2}nC)\}$
$= \pm \sin \tfrac{1}{2}nC$,
and $u = (1 - 2\sin^2 \tfrac{1}{2}nC) \pm 2\sin \tfrac{1}{2}nC \cos \tfrac{1}{2}n(A - B)$
$= 1 - 2\sin \tfrac{1}{2}nC\{\sin \tfrac{1}{2}nC \mp \cos \tfrac{1}{2}n(A - B)\}$
$= 1 - 2\sin \tfrac{1}{2}nC\{\pm \cos \tfrac{1}{2}n(A + B) \mp \cos \tfrac{1}{2}n(A - B)\}$
$= 1 - 2\sin \tfrac{1}{2}nC\{\mp 2\sin \tfrac{1}{2}nA \sin \tfrac{1}{2}nB\}$.

414. Let A_1, B_1, C_1, be the centres of the escribed circles opposite to the angles A, B, C, respectively, O of the inscribed circle, so that OA, OB, OC, will be perp. to B_1C_1, A_1C_1, and A_1B_1, respectively, and bisect the angles A, B, C:

then $2R = \frac{a}{\sin A}$; $2R_1 = \frac{a_1}{\sin A_1} = \frac{AB_1 + AC_1}{\sin A_1}$

$$= \frac{r_2 \sec \tfrac{1}{2}A + r_3 \sec \tfrac{1}{2}A}{\sin BOC} = \left(\frac{S}{s - b} + \frac{S}{s - c}\right) \sec^2 \tfrac{1}{2}A$$

(since $BOC = 180° - \tfrac{1}{2}B - \tfrac{1}{2}C = 90° + \tfrac{1}{2}A$)

$$= \frac{Sa}{(s - b)(s - c)} \sec^2 \tfrac{1}{2}A = a \sec^2 \tfrac{1}{2}A \sqrt{\frac{s(s - a)}{(s - b)(s - c)}}$$

$$= a \sec^2 \tfrac{1}{2}A \cot \tfrac{1}{2}A = \frac{2a}{\sin \mathrm{A}} = 4R;$$

$\therefore\ R_1 = 2R$, and so on.

15. From the first equation we get $\tan\alpha = \dfrac{1+h\cos\theta}{h\sin\theta}$,

and $1+h\cos\theta = h\sin\theta\tan\alpha$;

$\therefore\ \tan(\theta+\alpha) = \dfrac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha} =$ (substituting for $\tan\alpha$ and simplifying) $-\dfrac{h+\cos\theta}{\sin\theta}$;

hence $(1+h\cos\theta)\tan(\theta+\alpha) = h\sin\theta\tan\alpha \times -\dfrac{h+\cos\theta}{\sin\theta} =$ &c.

16. Since $2\cos\frac{1}{2}A = \sqrt{(2+2\cos A)}$,

$$\therefore\ 2\cos\frac{45^\circ}{2} = \sqrt{(2+2\cos 45^\circ)} = \sqrt{(2+\sqrt{2})};$$

$$\therefore\ \cos\frac{45^\circ}{2^2} = \sqrt{\left(2+2\cos\frac{45^\circ}{2}\right)} = \sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}, \text{ \&c.}$$

17. area of hexagon $AMCLBN$
$= \text{area}(ABC+ALC+AMC+ANB)$
$= S+\frac{1}{4}a^2+\frac{1}{4}b^2+\frac{1}{4}c^2$; but area $AMCLBN$
$=$ (also) area $(LMN+AMN+BLN+CLM)$

$$= S'+\tfrac{1}{2}\frac{b}{\sqrt{2}}\ \frac{c}{\sqrt{2}}\sin(90^\circ+A)+\text{\&c.}$$

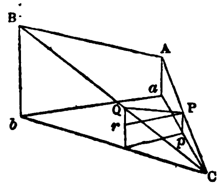
$$= S'+\tfrac{1}{4}(bc\cos A+ac\cos B+ab\cos C):$$

$$\therefore\ \frac{S'}{S} = 1+\frac{a^2+b^2+c^2-bc\cos A-ac\cos B-ab\cos C}{4S}$$

$$= 1+\frac{bc\cos A+ac\cos B+ab\cos C}{4S}$$

$$= 1+\tfrac{1}{2}\left\{\frac{\frac{1}{2}bc\cos A}{\frac{1}{2}bc\sin A}+\frac{\frac{1}{2}ac\cos B}{\frac{1}{2}ac\sin B}+\frac{\frac{1}{2}ab\cos C}{\frac{1}{2}ab\sin C}\right\} = 1+\tfrac{1}{2}p.$$

18.



Given $AB = AC = d$, $\angle aCb = \alpha$, $\angle ACa = \lambda$, $\angle BCb = \mu$: take equal lengths CP, CQ, each $= r$, in CA, CB, and project P, Q, in p, q, on the horizontal plane; draw Pr horizontal;

then distance $= 2d\cos ACB$.

now $PQ^2 = (2r\sin\frac{1}{2}ACB)^2 = 2r^2(1-\cos ACB) = Pr^2 + rQ^2$
$= pq^2 + (Qq - Pp)^2 = pq^2 + r^2(\sin\mu - \sin\lambda)^2$;
and $pq^2 = Cp^2 + Cq^2 - 2\,Cp\,.\,Cq\cos\alpha$
$= r^2(\cos^2\lambda + \cos^2\mu - 2\cos\lambda\cos\mu\cos\alpha)$;
$\therefore\ 2(1-\cos ACB)$
$= \cos^2\lambda + \cos^2\mu - 2\cos\lambda\cos\mu\cos\alpha + \sin^2\mu + \sin^2\lambda - 2\sin\lambda\sin\mu$
$= 2 - 2\sin\lambda\sin\mu - 2\cos\lambda\cos\mu\cos\alpha$;
$\therefore\ \cos ACB$
$= (\cos^2\frac{1}{2}\alpha + \sin^2\frac{1}{2}\alpha)\sin\lambda\sin\mu + (\cos^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\alpha)\cos\lambda\cos\mu$
$= \cos^2\frac{1}{2}\alpha\cos(\lambda-\mu) - \sin^2\frac{1}{2}\alpha\cos(\lambda+\mu)$: and distance $=$ &c.

419. Since $\cos\theta = \dfrac{a}{\sqrt{(a^2+b^2)}}$, $\sin\theta = \dfrac{b}{\sqrt{(a^2+b^2)}}$,

$\therefore\ a \pm b\sqrt{-1} = \sqrt{(a^2+b^2)}\{\cos\theta \pm \sqrt{-1}\sin\theta\}$;

and $\therefore\ u = (a^2+b^2)^{\frac{m}{2n}}\{(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}$

$+ (\cos\theta - \sqrt{-1}\sin\theta)^{\frac{m}{n}}\}$

$$= (a^2+b^2)^{\frac{m}{2n}}\left\{\left(\cos\frac{m}{n}\theta + \sqrt{-1}\sin\frac{m}{n}\theta\right)\right.$$
$$\left. + \left(\cos\frac{m}{n}\theta - \sqrt{-1}\sin\frac{m}{n}\theta\right)\right\}$$

$=$ &c.

420. Since $2\cos\alpha_n = x_n + \dfrac{1}{x_n}$,

$\therefore$ [13] $2\sqrt{-1}\sin\alpha_n = x_n - \dfrac{1}{x_n}$, and $x_n = \cos\alpha_n + \sqrt{-1}\sin\alpha_n$;

hence $x_1 x_2 \ldots x_n = (\cos\alpha_1 + \sqrt{-1}\sin\alpha_1)(\cos\alpha_2 + \sqrt{-1}\sin\alpha_2)\ldots$
$=$ by [1] $\cos(\alpha_1 + \alpha_2 + \ldots)$
$+ \sqrt{-1}\sin(\alpha_1 + \alpha_2 + \ldots)$
$= \cos 2\pi + \sqrt{-1}\sin 2\pi = 1$.

421. $(a \pm b)^2 = 2(1 \pm \sqrt{-1}\tan 2\phi) = 2\sec 2\phi(\cos 2\phi \pm \sqrt{-1}\sin 2\phi)$;
$\therefore\ a \pm b =$ by [3] $\sqrt{(2\sec 2\phi)}(\cos\phi \pm \sqrt{-1}\sin\phi)$,
and, adding these, $2a =$ &c.

422. $\theta = \cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 -$ &c.;
$\therefore\ \theta^2 + 2\theta - 2 = \frac{1}{12}\theta^4 -$ &c. $= 0$ for 1st app.:
hence $\theta + 1 = \sqrt{3}$, and $\theta = \sqrt{3} - 1 = .7320508$:
$\therefore\ \theta^2 + 2\theta - 2 = \frac{1}{12}\theta^4$, for 2nd app., $= \frac{1}{12}(\sqrt{3}-1)^4$
$= \frac{1}{3}(7 - 4\sqrt{3}) = .023932257$;

$$\therefore\ \theta^2+2\theta+1=3.023932257, \text{ and } \theta+1=1.738945,$$
$$\text{or } \theta=.738945=(\text{in degrees})\ .738945\times 57^\circ.29577$$
$$=42^\circ 20' 18''.$$

23. $$\tan^{-1}a+\tan^{-1}\frac{a}{1+1.2a^2}=\tan^{-1}\frac{a(1+2a^2)+a}{(1+2a^2)-a}=\tan^{-1}2a;$$
$$\tan^{-1}2a+\tan^{-1}\frac{a}{1+2.3a^2}=\tan^{-1}\frac{2a(1+2.3a^2)+a}{(1+2.3a^2)-2a^2}$$
$$=\tan^{-1}3a;\ \&c.=\&c.$$
$$\tan^{-1}(n-1)a+\tan^{-1}\frac{a}{1+(n-1)na^2}$$
$$=\tan^{-1}\frac{(n-1)a\{1+(n-1)na^2\}+a}{\{1+(n-1)na^2\}-(n-1)a^2}=\tan^{-1}na.$$

24. Given *line* $BC=c$, *arc* $BC=l$:
$$\text{now } \frac{c}{r}=\text{chord}\frac{l}{r}=2\sin\frac{l}{2r}=2\left\{\frac{l}{2r}-\frac{1}{1.2.3}\left(\frac{l}{2r}\right)^3\right\}, \text{ nearly,}$$
by [6]:
$$\therefore\ c=l-\frac{l^3}{24r^2}, \text{ and } r=\&c.$$

25. $$2S=(1-\cos 2\theta)+(1-\cos 4\theta)+\&c.$$
$$=n-(\cos 2\theta+\cos 4\theta+\&c.)$$
$$=n-\frac{\cos\{2\theta+(n-1)\theta\}\sin n\theta}{\sin\theta} \text{ by [39 Cor.]}$$
$$=n-\frac{\sin n\theta\cos(n+1)\theta}{\sin\theta};$$
$$\therefore\ S=\tfrac{1}{2}\left\{n-\frac{\sin(2n+1)\theta-\sin\theta}{2\sin\theta}\right\}=\&c.$$

26. By [30] $$\sin\theta=2^{n-1}\sin\frac{\theta}{n}\sin\frac{\pi+\theta}{n}\sin\frac{2\pi+\theta}{n}$$
$$\ldots\sin\frac{(n-1)\pi+\theta}{n}:$$
put $\theta=\frac{1}{2}\pi$, $n=18$; then $1=2^{17}.\sin 5^\circ\sin 15^\circ\ldots\sin 175^\circ$
$=2^{17}.\sin 5^\circ\sin 15^\circ\ldots\sin 85^\circ\sin 85^\circ\ldots\sin 15^\circ\sin 5^\circ$;
$$\text{or } \sin 5^\circ\sin 15^\circ\ldots\sin 85^\circ=\frac{1}{\sqrt{2^{17}}}=\frac{1}{256\sqrt{2}}.$$

27. Proceeding as in [21], we shall find $\cos a=-\frac{1}{2}$, or $a=120^\circ$, and $360^\circ\pm 120^\circ$; hence the values of x are $2\cos 40^\circ$, $2\cos 160^\circ$, $2\cos 80^\circ$, or $2\cos 40^\circ$, $-2\cos 20^\circ$, $2\sin 10^\circ$.

28. Giving n the values 1, 2, 3, &c., we get the series
$$1-\tfrac{2}{3}.\frac{1}{1.2}+\tfrac{3}{5}.\frac{1}{1.2.3.4}-\&c=1-\frac{2}{1.2.3}+\frac{3}{1.2.3.4.5}-\&c.$$

Now $\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$,

and $x \cos x = x - \frac{x^3}{1.2} + \frac{x^5}{1.2.3.4} - \&c.$

$\therefore \sin x + x \cos x = 2x - \frac{4x^3}{1.2.3} + \frac{6x^5}{1.2.3.4.5} - \&c. = 2S$, if $x = 1$;

hence $S = \frac{1}{2}(\sin 1 + \cos 1) = \frac{1}{\sqrt{2}}\left(\sin 1 \cos\frac{\pi}{4} + \cos 1 \sin\frac{\pi}{4}\right) = \&c.$

429. Since $1 = \cot\frac{1}{4}\pi = \cot .7854$, it is plain that $\theta = \frac{1}{4}\pi + \phi$, where ϕ is small; and $\theta = \cot\theta$,

or $\frac{1}{4}\pi + \phi = \frac{1 - \tan\phi}{1 + \tan\phi} = 1 - \frac{2\tan\phi}{1 + \tan\phi}$

$= 1 - 2\tan\phi + 2\tan^2\phi - 2\tan^3\phi + \&c.$

$= [6]\ 1 - 2(\phi + \frac{1}{3}\phi^3) + 2(\phi^2 + \frac{2}{3}\phi^4) - 2\phi^3 + 2\phi^4$,

as far as terms involving ϕ^4:

hence $1 - \frac{1}{4}\pi = x\,(\text{suppose}) = 3\phi - 2\phi^2 + \frac{8}{3}\phi^3 - \frac{10}{3}\phi^4 + \&c.$

Assume $\phi = \frac{1}{3}x + ax^2 + bx^3 + \&c.$;

then $x = x + 3ax^2 + 3bx^3 + \&c.$

$- \frac{2}{9}x^2 - \frac{4}{3}ax^3 - \&c.$

$+ \frac{8}{81}x^3 - \&c.\ \therefore\ a = \frac{2}{27},\ b = 0$;

and $\phi = \frac{1}{3}x + \frac{2}{27}x^2 = \frac{1}{3}(.2146) + \frac{2}{27}(.2146)^2 = .0748$,

and $\theta = \frac{1}{4}\pi + \phi = .7854 + .0748 = .8602$

$=$ (in degrees) 49° 17′, nearly.

430. $e^x \cos x = \frac{1}{2}e^x(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}) = \frac{1}{2}\{e^{x(1+\sqrt{-1})} + e^{x(1-\sqrt{-1})}\}$,

where the coeff. of $x^n = \frac{1}{2}\{(1 + \sqrt{-1})^n + (1 - \sqrt{-1})^n\}$

$= \frac{1}{2}k^n\{(\cos\theta + \sqrt{-1}\sin\theta)^n + (\cos\theta - \sqrt{-1}\sin\theta)^n\}$,

if $k\cos\theta = 1$, $k\sin\theta = 1$, whence $k^2 = 2$, and $\tan\theta = 1$, or $\theta = \frac{1}{4}\pi$;

hence coeff. of $x^n = \frac{1}{2}.2^{\frac{1}{2}n}\{(\cos n\vartheta + \sqrt{-1}\sin n\theta)$

$+ (\cos n\theta - \sqrt{-1}\sin n\vartheta)\}$

$= \frac{1}{2}.2^{\frac{1}{2}n}.2\cos n\theta = 2^{\frac{1}{2}n}\cos\frac{1}{4}n\pi.$

431. By [30 Cor.] $1 = 2^{n-1}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\sin\frac{5\pi}{2n}\ldots\sin\frac{(2n-1)\pi}{2n}$:

put $n = 9$;

then $1 = 2^8 \sin 10° \sin 30° \sin 50° \ldots \sin 170°$

$= 2^8 \sin 10° \sin 30° \ldots \sin 70° \sin 70° \ldots \sin 30° \sin 10°$

$= (2^4 \sin 10° \sin 30° \sin 50° \sin 70°)^2.$

432. $x^3 - \frac{3}{2}x - \frac{1}{2} = 0$;

hence [21] $\cos a = \frac{1}{4} \times 2^{\frac{3}{2}} = \frac{1}{\sqrt{2}}$, and $\alpha = 45°$, or $360° \pm 45°$;

$\therefore\ x = 2\sqrt{\tfrac{1}{2}}\cos 15^\circ = \tfrac{1}{2}(\sqrt{3}+1)$, or $\sqrt{2}\cos 135^\circ = -1$,
or $\sqrt{2}\cos 105^\circ = -\sqrt{2}\sin 15^\circ = -\tfrac{1}{2}(\sqrt{3}-1)$.

133. If $x = e^{a\sqrt{-1}}$ and $y = e^{\beta\sqrt{-1}}$,

then, by [7] and question, $\dfrac{x^2-1}{x^2+1} = \dfrac{m(y^2-1)}{y^2+1}$;

$$\therefore\ x^2 = \frac{y^2+1+m(y^2-1)}{y^2+1-m(y^2-1)} = \frac{(m+1)y^2-(m-1)}{(m+1)-(m-1)y^2} = y^2\frac{1-py^{-2}}{1-py^2},$$

if $p = \dfrac{m-1}{m+1}$;

$$\therefore\ 2\log_e x = 2\log_e y + \log_e(1-py^{-2}) - \log_e(1-py^2),$$

$$\text{or } 2a\sqrt{-1} = 2\beta\sqrt{-1} - (py^{-2} + \tfrac{1}{2}p^2y^{-4} + \&c.)$$
$$+ (py^2 + \tfrac{1}{2}p^2y^4 + \&c.):$$

$$\therefore\ a = \beta + p\left(\frac{y^2-y^{-2}}{2\sqrt{-1}}\right) + \frac{p^2}{2}\left(\frac{y^4-y^{-4}}{2\sqrt{-1}}\right) + \&c. = \&c.$$

34. Let $2\sqrt{-1}\sin\theta = y - \dfrac{1}{y}$;

$$\text{then } 2\sqrt{-1}\Sigma = x\left(y-\frac{1}{y}\right) + \tfrac{1}{2}x^2\left(y^2-\frac{1}{y^2}\right) + \&c.$$

$$= (xy + \tfrac{1}{2}x^2y^2 + \&c.) - \left(\frac{x}{y} + \tfrac{1}{2}\frac{x^2}{y^2} + \&c.\right)$$

$$= -\log_e(1-xy) + \log_e\left(1-\frac{x}{y}\right) = \log^e\frac{1-\dfrac{x}{y}}{1-xy}:$$

$$\therefore\ e^{2\Sigma\sqrt{-1}} = \frac{1-\dfrac{x}{y}}{1-xy}, \text{ and } \tan\Sigma = \frac{1}{\sqrt{-1}}\cdot\frac{e^{2\Sigma\sqrt{-1}}-1}{e^{2\Sigma\sqrt{-1}}+1} \text{ by [7]}$$

$$= \frac{1}{\sqrt{-1}}\cdot\frac{x\left(y-\dfrac{1}{y}\right)}{2-x\left(y+\dfrac{1}{y}\right)} = \frac{1}{\sqrt{-1}}\ \frac{x(2\sqrt{-1}\sin\theta)}{2-x(2\cos\theta)}$$

$$= \frac{x\sin\theta}{1-x\cos\theta}.$$

35. Put $y + \tfrac{3}{2}$ for x; then, simplifying, we get $y^3 - \tfrac{3}{2}y - \tfrac{1}{2} = 0$;

whence [21] $\cos a = \dfrac{1}{\sqrt{2}}$, and $a = 45^\circ$ or $360^\circ \pm 45^\circ$,

and $y = \sqrt{2}\cos\tfrac{1}{3}a = \sqrt{2}\cos 15^\circ$, or $\sqrt{2}\cos 135^\circ$, or $\sqrt{2}\cos 105^\circ$,
$= \tfrac{1}{2}(\sqrt{3}+1)$, -1, or $-\tfrac{1}{2}(\sqrt{3}-1)$;
and $x = y + \tfrac{3}{2} = \&c.$

436. In [30 (3)] write m for n, and put $\theta = \frac{1}{4}\pi$;

$$\text{then } \tan\frac{\pi}{4m}\tan\frac{5\pi}{4m}\ldots\tan\frac{(4m-3)\pi}{4m} = (m \text{ even}) + 1,$$

or $= (m \text{ odd}) \pm \tan\theta = (\text{also}) \pm 1$.

437. $\cos 2\theta \operatorname{cosec}^2 2\theta = \dfrac{2\cos^2\theta - 1}{4\sin^2\theta\cos^2\theta} = \frac{1}{2}\operatorname{cosec}^2\theta - \operatorname{cosec}^2 2\theta$;

so $2\cos 4\theta \operatorname{cosec}^2 4\theta = \operatorname{cosec}^2 2\theta - 2\operatorname{cosec}^2 2^2\theta$,

$2^2\cos 8\theta \operatorname{cosec}^2 8\theta = 2\operatorname{cosec}^2 2^2\theta - 2^2\operatorname{cosec}^2 2^3\theta$, &c.

$$\therefore S = \tfrac{1}{2}\operatorname{cosec}^2\theta - 2^{n-1}\operatorname{cosec}^2 2^n\theta.$$

438. $\tan^2\frac{1}{4}a = \dfrac{\tan(\frac{1}{4}a - \phi)}{\tan\frac{1}{4}a}$,

and $\therefore \cos\frac{1}{2}a = \dfrac{1 - \tan^2\frac{1}{4}a}{1 + \tan^2\frac{1}{4}a} = \dfrac{\sin\phi}{\sin(\frac{1}{2}a - \phi)}$,

whence $\sin\frac{1}{2}a\cot\phi = \cos\frac{1}{2}a + \sec\frac{1}{2}a$,

or $\tan\phi = \dfrac{\sin a}{2 + 2\cos^2\frac{1}{2}a} = \dfrac{\sin a}{3 + \cos a}$:

whence, putting $2\cos\phi = x + \dfrac{1}{x}$, so that [13] $x = e^{\phi\sqrt{-1}}$,

and $2\cos a = y + \dfrac{1}{y}$, we have [13] $\dfrac{x^2 - 1}{x^2 + 1} = \dfrac{y - \frac{1}{y}}{6 + y + \frac{1}{y}}$,

or $x^2 = \dfrac{6 + 2y}{6 + 2y^{-1}} = \dfrac{1 + \frac{1}{3}y}{1 + \frac{1}{3}y^{-1}}$;

$$\therefore 2\log_e x = \left(\tfrac{1}{3}y - \frac{1}{2.3^2}y^2 + \&\text{c.}\right) - \left(\tfrac{1}{3}y^{-1} - \frac{1}{2.3^2}y^{-2} + \&\text{c.}\right)$$

or $2\phi\sqrt{-1} = \frac{1}{3}(y - y^{-1}) - \dfrac{1}{2.3^2}(y^2 - y^{-2}) + \&\text{c.}$;

$$\therefore \phi = \tfrac{1}{3}\sin a - \frac{1}{2.3^2}\sin 2a + \&\text{c.}$$

439. Since $4\sin^3\theta = 3\sin\theta - \sin 3\theta$;

$\therefore 4S = 3(\sin\theta + \sin 2\theta + \&\text{c.}) - (\sin 3\theta + \sin 6\theta + \&\text{c.})$

$$= [39]\ 3\frac{\sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta} - \frac{\sin\frac{3}{2}(n+1)\theta\sin\frac{3}{2}n\theta}{\sin\frac{3}{2}\theta}.$$

440. $\dfrac{b}{a} = \dfrac{\sin B}{\sin A} = \dfrac{y - y^{-1}}{x - x^{-1}}$, if $2\cos A = x + \dfrac{1}{x}$, $2\cos B = y + \dfrac{1}{y}$;

$$\therefore \log_e\frac{b}{a} = \tfrac{1}{2}\log_e\left(\frac{y - y^{-1}}{x - x^{-1}}\right)^2 = \tfrac{1}{2}\log_e\frac{y^2 - 2 + y^{-2}}{x^2 - 2 + x^{-2}}$$

$$= \tfrac{1}{2}\log_e\frac{(1 - y^2)(1 - y^{-2})}{(1 - x^2)(1 - x^{-2})}$$

$$=\tfrac{1}{2}\{\log_e(1-y^2)+\log_e(1-y^{-2})-\log_e(1-x^2)$$
$$-\log_e(1-x^{-2})\}$$
$$=\tfrac{1}{2}\{-(y^2+\tfrac{1}{2}y^4+\&c.)-(y^{-2}+\tfrac{1}{2}y^{-4}+\&c.)$$
$$+(x^2+\tfrac{1}{2}x^4+\&c.)+(x^{-2}+\tfrac{1}{2}x^{-4}+\&c.)\}$$
$$=\tfrac{1}{2}\{\{(x^2+x^{-2})-(y^2+y^{-2})\}$$
$$+\tfrac{1}{2}\{(x^4+x^{-4})-(y^4+y^{-4})\}+\&c.\}$$
$$=(\cos 2A-\cos 2B)+\tfrac{1}{2}(\cos 4A-\cos 4B)+\&c.$$

441. By [13] $S=x+x^2+\&c.,\left(\text{if } 2\cos\alpha=x+\frac{1}{x},\right)$

$$=x\frac{x^n-1}{x-1}=x^{\frac{n+1}{2}}\cdot\frac{x^{\frac{1}{2}n}-x^{-\frac{1}{2}n}}{x^{\frac{1}{2}}-x^{-\frac{1}{2}}}$$

$$=\{\cos\tfrac{1}{2}(n+1)\alpha+\sqrt{-1}\sin\tfrac{1}{2}(n+1)\alpha\}\frac{\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}:$$

but $S=(\text{also})\,(\cos\alpha+\sqrt{-1}\sin\alpha)+(\cos 2\alpha+\sqrt{-1}\sin 2\alpha)$
$+\&c.;$

$\therefore$, equating possible and impossible parts,

$\cos\alpha+\cos 2\alpha+\&c.=\dfrac{\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}\cos\tfrac{1}{2}(n+1)\alpha,$

$\sin\alpha+\sin 2\alpha+\&c.=\&c.$

442. By [39] $S=\dfrac{\cos\{A+(n-1)A\}\sin nA}{\sin A}=\tfrac{1}{2}\sin 2nA\operatorname{cosec} A.$

443. $\tan\theta=\dfrac{\sin A\sin B}{1-\sin A\cos B}$, or $\dfrac{y^2-1}{y^2+1}=\dfrac{\sin A(x-x^{-1})}{2-\sin A(x+x^{-1})}$,

if $2\cos\theta=y+\dfrac{1}{y}$, $2\cos B=x+\dfrac{1}{x}$;

$\therefore\ y^2=\dfrac{1-x^{-1}\sin A}{1-x\sin A}$,

$$\text{and } 2\log_e y=2\theta\sqrt{-1}$$
$$=\log_e(1-x^{-1}\sin A)-\log_e(1-x\sin A)$$
$$=(x\sin A+\tfrac{1}{2}x^2\sin^2 A+\&c.)$$
$$-(x^{-1}\sin A+\tfrac{1}{2}x^{-2}\sin^2 A+\&c.)$$
$$=(x-x^{-1})\sin A+\tfrac{1}{2}(x^2-x^{-2})\sin^2 A+\&c.;$$

$\therefore\ \theta=\sin B\sin A+\tfrac{1}{2}\sin 2B\sin^2 A+\&c.$

444. Put $2\cos\theta=x+x^{-1}$;

then $2\Sigma=(x+x^{-1})+\dfrac{\cos\theta}{1}(x^2+x^{-2})+\dfrac{\cos^2\theta}{1.2}(x^3+x^{-3})+\&c.$

$$=x\left(1+x\cos\theta+\frac{x^2\cos^2\theta}{1.2}+\&c.\right)$$

$$+x^{-1}\left(1+x^{-1}\cos\theta+\frac{x^{-2}\cos^2\theta}{1.2}+\&c.\right)=\&c., \text{ by (146).}$$

445. Since $2\cot 2A=\dfrac{\cot^2 A-1}{\cot A}=\cot A-\tan A$,

$$\therefore \cot A=\tan A+2\cot 2A$$
$$=\tan A+2\tan 2A+2^2\cot 2^2 A=\&c.:$$

$$\therefore \cot\frac{\pi}{2^{x+1}}=\tan\frac{\pi}{2^{x+1}}+2\tan\frac{\pi}{2^x}+2^2\tan\frac{\pi}{2^{x-1}}+\&c.$$
$$+2^{x-2}\tan\frac{\pi}{2^3}+2^{x-1}\cot\frac{\pi}{4};$$

$$\text{or } 1=\tan\frac{\pi}{2^{x+1}}\left\{\tan\frac{\pi}{2^{x+1}}+\&c.+2^{x-1}\right\}.$$

446. $1-n\cos\theta=1-\frac{1}{2}n(x+x^{-1})$, suppose, $=(a-\beta x)(a-\beta x^{-1})$, if $1=a^2+\beta^2$, and $\frac{1}{2}n=a\beta$, whence $(a+\beta)^2=1+n$, and a, β, are roots of $x^2-2x\sqrt{(1+n)}+\frac{1}{2}n=0$:

$$\therefore \log_e(1-n\cos\theta)$$
$$=\log_e\{a^2(1-mx)(1-mx^{-1})\}$$
$$=2\log_e a+\log_e(1-mx)+\log_e(1-mx^{-1})$$
$$=2\log_e a-\{m(x+x^{-1})+\tfrac{1}{2}m^2(x^2+x^{-2})+\&c.\}$$
$$=2(\log_e a-m\cos\theta-\tfrac{1}{2}m^2\cos 2\theta-\&c.)$$

447. Let $\tan\theta=x+h$, $\tan\phi=x$;

then $\tan(\theta-\phi)=\dfrac{h}{1+x(x+h)}=\dfrac{1}{\sqrt{-1}}\cdot\dfrac{y^2-1}{y^2+1}$,

if $y=e^{(\theta-\phi)\sqrt{-1}}$;

$$\therefore y^2=\frac{1+x^2+h(x+\sqrt{-1})}{1+x^2+h(x-\sqrt{-1})}=\frac{\operatorname{cosec}^2 z+h(\cot z+\sqrt{-1})}{\operatorname{cosec}^2 z+h(\cot z-\sqrt{-1})}$$
$$=\frac{1+h\sin z(\cos z+\sqrt{-1}\sin z)}{1+h\sin z(\cos z-\sqrt{-1}\sin z)}=\frac{1+h\sin z.v}{1+h\sin z.v^{-1}},$$

if $v=\cos z+\sqrt{-1}\sin z=e^{z\sqrt{-1}}$;

$$\therefore 2\log_e y=\log_e(1+hv\sin z)-\log_e(1+hv^{-1}\sin z),$$
$$\text{or } 2(\theta-\phi)\sqrt{-1}=h\sin z(v-v^{-1})-\tfrac{1}{2}h^2\sin^2 z(v^2-v^{-2})+\&c.$$
$$\text{or } \theta-\phi=h\sin z(\sin z)-\tfrac{1}{2}h^2\sin^2 z(\sin 2z)+\&c.$$

448. The general term is $\dfrac{\sin 2n\theta}{\cos(2n-1)\theta\cos(2n+1)\theta}$

$$=\left\{\frac{1}{\cos(2n+1)\theta}-\frac{1}{\cos(2n-1)\theta}\right\}\frac{1}{2\sin\theta};$$

$$\therefore S=\frac{1}{2\sin\theta}\left\{\left(\frac{1}{\cos 3\theta}-\frac{1}{\cos\theta}\right)+\left(\frac{1}{\cos 5\theta}-\frac{1}{\cos 3\theta}\right)+\&c.\right.$$
$$\left.+\left(\frac{1}{\cos(2n+1)\theta}-\frac{1}{\cos(2n-1)\theta}\right)\right\}$$

$$=\frac{1}{2\sin\theta}\left(\frac{1}{\cos(2n+1)\theta}-\frac{1}{\cos\theta}\right)=\&c.$$

49. Put $2\cos\theta=y+\frac{1}{y}$; then we have $\frac{y^4-1}{y^4+1}=\sqrt{-1}\cos\alpha\tan 2\phi$,

$$\text{or } y^4=\frac{1+\sqrt{-1}\cos\alpha\tan 2\phi}{1-\sqrt{-1}\cos\alpha\tan 2\phi}$$

$$=\frac{(1-\tan^2\phi)+\sqrt{-1}\,(x+x^{-1})\tan\phi}{(1-\tan^2\phi)-\sqrt{-1}\,(x+x^{-1})\tan\phi},\text{ if } 2\cos\alpha=x+\frac{1}{x},$$

$$=\frac{(1+\sqrt{-1}\,x\tan\phi)(1+\sqrt{-1}\,x^{-1}\tan\phi)}{(1-\sqrt{-1}\,x\tan\phi)(1-\sqrt{-1}\,x^{-1}\tan\phi)}:$$

$$4\log_e y=\text{by (150) } 2\{\sqrt{-1}\,x\tan\phi-\tfrac{1}{3}x^3\tan^3\phi+\&c.\}$$
$$+2\{\sqrt{-1}\,x^{-1}\tan\phi-\tfrac{1}{3}x^{-3}\tan^3\phi+\&c.\};$$

or $4\theta\sqrt{-1}=2\sqrt{-1}\{(x+x^{-1})\tan\phi-\frac{1}{3}(x^3+x^{-3})\tan^3\phi+\&c.\}$;

$\therefore\ \theta=\cos\alpha\tan\phi-\frac{1}{3}\cos 3\alpha\tan^3\phi+\&c.$

50. $S=2r\sin\frac{\pi}{n}+2r\sin\frac{2\pi}{n}+\&c.$, or, putting a for $\frac{\pi}{n}$,

$=2r(\sin a+\sin 2a+\&c.$ to $(n-1)$ terms),

$$=[39]\,2r\frac{\sin\{a+\frac{1}{2}(n-2)a\}\sin\frac{1}{2}(n-1)a}{\sin\frac{1}{2}a}$$

$$=2r\sin\frac{\pi}{2}\times\frac{\sin\frac{(n-1)\pi}{2n}}{\sin\frac{\pi}{2n}}=\frac{a}{\sin\frac{\pi}{n}}\times\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$$

$$=\frac{a}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}\times\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}=\tfrac{1}{2}a\,\mathrm{cosec}^2\frac{\pi}{2n}.$$

51. $u=\log_e\{\frac{1}{2}(a+b)+\frac{1}{2}(a-b)\cos 2\theta\}$

$=\log_e\{\frac{1}{2}(a+b)+\frac{1}{4}(a-b)(x^2+x^{-2})\}$

$=\log_e\{(m+nx^2)(m+nx^{-2})\}$, if $m^2+n^2=\frac{1}{2}(a+b)$,

$mn=\frac{1}{4}(a-b)$, whence $(m+n)^2=a, (m-n)^2=b$,

and $m=\frac{1}{2}(\sqrt{a}+\sqrt{b})$, $n=\frac{1}{2}(\sqrt{a}-\sqrt{b})$:

now $u=\log_e m^2+\log_e\left(1+\frac{n}{m}x^2\right)+\log_e\left(1+\frac{n}{m}x^{-2}\right)$

$$=2\log_e m+\frac{n}{m}(x^2+x^{-2})-\frac{n^2}{2m^2}(x^4+x^{-4})+\&c.$$

$$=2\log_e m+\frac{n}{m}(2\cos 2\theta)-\frac{n^2}{2m^2}(2\cos 4\theta)+\&c.$$

$$=2\log_e m+\frac{n}{m}(4\cos^2\theta-2)-\frac{n^2}{2m^2}(4\cos^2 2\theta-2)+\&c.$$

$$=2\log_e m-2\left(\frac{n}{m}-\frac{n^2}{2m^2}+\&c.\right)$$

$$+4\left(\frac{n}{m}\cos^2\theta-\frac{n^2}{2m^2}\cos^2 2\theta+\&c.\right)$$

$$=2\log_e m-2\log_e\left(1+\frac{m}{n}\right)+\&c.=2\log_e\frac{m^2}{m+n}+\&c.$$

$$=2\log_e\frac{(\sqrt{a}+\sqrt{b})^2}{4\sqrt{a}}+\&c.$$

$$=4\log_e\frac{\sqrt{a}+\sqrt{b}}{2^4 a}+4\left(\frac{n}{m}\cos^2\theta-\&c.\right),$$

where $\frac{n}{m}=\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

452. Let OA be the radius of the inner circle, radius r, from which the divisions begin at the points P_1, P_2, &c.; and let B be the point in outer circle, radius R, and let $\angle AOB=a$;

then $BP_1{}^2=R^2+r^2-2Rr\cos\left(\frac{2\pi}{n}-a\right)$,

$$BP_2{}^2=R^2+r^2-2Rr\cos\left(\frac{4\pi}{n}-a\right),\ \&c.=\&c.$$

$$\therefore\ S=n(R^2+r^2)-2Rr\{\cos(\theta-a)+\cos 2(\theta-a)\&c.\},$$

if $\theta=\frac{2\pi}{n}$,

$=n(R^2+r^2)$, since [39] the numerator of the series involves the factor $\sin\frac{1}{2}n\theta=\sin\pi=0$.

453. $2\sqrt{-1}\,S=(x-x^{-1})\sin\theta-\frac{1}{2}(x^2-x^{-2})\sin^2\theta+\&c.$

$$=(x\sin\theta-\tfrac{1}{2}x^2\sin^2\theta+\&c.)$$

$$-(x^{-1}\sin\theta-\tfrac{1}{2}x^{-2}\sin^2\theta+\&c.)$$

$$=\log_e(1+x\sin\theta)-\log_e(1+x^{-1}\sin\theta)$$

$$=\log_e\frac{1+x\sin\theta}{1+x^{-1}\sin\theta}:$$

$$\therefore\ \cot S=\sqrt{-1}\ \frac{e^{2S\sqrt{-1}}+1}{e^{2S\sqrt{-1}}-1}$$

$$=\sqrt{-1}\,.\,\frac{(1+x\sin\theta)+(1+x^{-1}\sin\theta)}{(1+x\sin\theta)-(1+x^{-1}\sin\theta)}$$

$$=\sqrt{-1}\,.\,\frac{2+(x+x^{-1})\sin\theta}{(x-x^{-1})\sin\theta}=\frac{1+\cos\theta\sin\theta}{\sin\theta}$$

$$=\frac{(\sin^2\theta+\cos^2\theta)+\cos\theta\sin\theta}{\sin^2\theta}=\&c.$$

154. By [19] $\cos n\theta = 1 - \frac{n^2}{1.2}\sin^2\theta + \text{\&c.} + (-1)^{\frac{1}{2}n}2^{n-1}\sin^n\theta$;

put $\cos n\theta = 0$; then $\theta = \pm\frac{\pi}{2n}, \pm\frac{3\pi}{2n}, \text{\&c.} \pm\frac{(n-1)\pi}{2n}$,

and the equation becomes $0 = 1 - \frac{n^2}{1.2}\sin^2\theta + \text{\&c.}$,

or $\operatorname{cosec}^n\theta - \frac{n^2}{1.2}\operatorname{cosec}^{n-2}\theta + \text{\&c.} + (-1)^{\frac{1}{2}n}2^{n-1} = 0$;

$\therefore$, since the roots of this equation are $\operatorname{cosec}^2\frac{\pi}{2n}, \operatorname{cosec}^2\frac{3\pi}{2n}$, &c., we have, by Theory of Equations,

$$\operatorname{cosec}^2\frac{\pi}{2n} + \text{\&c.} = \text{sum of roots} = \text{coeff. of 2nd term} = \frac{n^2}{2}.$$

55. Let the diameter AB be divided by the lines in P_1, P_2, &c. then (O being the centre), we have

$$CP_1 = CO\sec OCP_1 = r\sec\left(\frac{n-1}{n+1}\cdot\frac{\pi}{4}\right),$$

$$CP_2 = r\sec\left(\frac{n-3}{n+1}\cdot\frac{\pi}{4}\right), \text{ \&c.}$$

$$\therefore S = \frac{1}{r^2}\{\cos^2 a + \cos^2(a-\theta) + \text{\&c. to } n \text{ terms}\},$$

$$\text{where } a = \frac{n-1}{n+1}\cdot\frac{\pi}{4}, \theta = \frac{\pi}{2(n+1)},$$

$$= \frac{1}{2r^2}\{n + \cos 2a + \cos 2(a-\theta) + \text{\&c.}\}$$

$$= [39]\,\frac{1}{2r^2}\left\{n + \frac{\cos\{2a-(n-1)\theta\}\sin(-n\theta)}{\sin(-\theta)}\right\}$$

$$= \frac{1}{2r^2}\left\{n + \frac{\sin\frac{n\pi}{2(n+1)}}{\sin\theta}\right\} = \frac{1}{2r^2}\left\{n + \frac{\cos\theta}{\sin\theta}\right\}.$$

56. Let P_1, P_2, &c., be the points of the circle, and let OC produced cut the circle in A: then if $\angle AOP_1 = a$,

and $\angle P_1OP_2 = P_2OP_3 = \text{\&c.} = \frac{2\pi}{n} = \theta$,

we have $a^2 = b^2 + r_1^2 - 2br_1\cos a$,

$$a^2 = b^2 + r_2^2 - 2br_2\cos(a+\theta), \text{ \&c.},$$

or $(a^2-b^2)\frac{1}{r_1} = r_1 - 2b\cos a$,

$$(a^2-b^2)\frac{1}{r_2} = r_2 - 2b\cos(\alpha+\theta), \text{ \&c.} \therefore$$

$$\therefore\ r_1 + r_2 + \&c. = (a^2 - b^2)\left(\frac{1}{r_1} + \frac{1}{r_2} + \&c.\right)$$
$$+ 2b\{\cos a + \cos(a+\theta) + \&c.\},$$

where the sum of the latter series $= 0$, since [39] it contains in its numerator the factor $\sin\frac{1}{2}n\theta = \sin\pi = 0$.

457. Let $AP_1P_2 \ldots P_{n-1}$ be the polygon, A being the given angular point; then (115) radius of circle in triangle

$$AP_{r-1}P_r = \frac{\text{area}\,AP_{r-1}P_r}{s} = \frac{\frac{1}{2}AP_{r-1}\,.\,AP_r \sin\theta}{\frac{1}{2}(AP_{r-1} + AP_r + P_{r-1}P)},$$

$\left(\text{where } \theta = \frac{\pi}{n} = \angle \text{ at circumference upon any one of the sides;}\right)$

now $P_1P_2 = 2r\sin\theta = P_2P_3 = \ldots = P_{r-1}P_r = \ldots = P_{n-1}A$,
and $AP_1 = (\text{chord } 2\theta =) 2r\sin\theta$,
$AP_2 = 2r\sin 2\theta, \ldots AP_{n-1} = 2r\sin(n-1)\theta$:

$$\therefore \text{ radius of circle in } AP_{r-1}P_r = 2r\frac{\sin(r-1)\theta\sin r\theta\sin\theta}{\sin(r-1)\theta + \sin r\theta + \sin\theta}$$

$$= 4r\sin\tfrac{1}{2}(r-1)\theta\cos\tfrac{1}{2}r\theta\sin\tfrac{1}{2}\theta;$$

$$\therefore \text{ sum of areas of circles} = 16\pi r^2\sin^2\tfrac{1}{2}\theta\left\{\sin^2\frac{\theta}{2}\,.\cos^2\frac{2\theta}{2}\right.$$
$$\left. + \sin^2\frac{2\theta}{2}\,.\,\cos^2\frac{3\theta}{2} + \ldots\sin^2\frac{(n-1)\theta}{2}\cos^2\frac{n\theta}{2}\right\},$$

where the series

$$= \tfrac{1}{4}\{(1-\cos\theta)(1+\cos 2\theta) + (1-\cos 2\theta)(1+\cos 3\theta) + \&c.\}$$
$$= \tfrac{1}{4}\{(n-1) - \cos\theta + \cos n\theta - (\cos\theta\cos 2\theta + \cos 2\theta\cos 3\theta + \&c.)\}$$
$$= \tfrac{1}{4}\{(n-2) - \cos\theta - \tfrac{1}{2}\{(\cos\theta + \cos 3\theta) + (\cos\theta + \cos 5\theta) + \&c.\}\},$$

since $n\theta = \pi$,

$$= \tfrac{1}{4}\left\{n - 2 - \cos\theta - \tfrac{1}{2}\left\{(n-1)\cos\theta\right.\right.$$
$$\left.\left. + \frac{\cos\{3\theta + (n-2)\theta\}\sin(n-1)\theta}{\sin\theta}\right\}\right\}$$

$$= \tfrac{1}{4}\left\{n - 2 - \tfrac{1}{2}(n+1)\cos\theta - \frac{\cos(\pi+\theta)\sin(\pi-\theta)}{2\sin\theta}\right\}$$

$$= \tfrac{1}{4}\{n - 2 - \tfrac{1}{2}n\cos\theta\} = \tfrac{1}{4}\{n - 2 - \tfrac{1}{2}n(1 - 2\sin^2\tfrac{1}{2}\theta)\}$$
$$= \tfrac{1}{8}(n-4) + \tfrac{1}{4}n\sin^2\tfrac{1}{2}\theta.$$

THE END.

LONDON: SPOTTISWOODES and SHAW, New-street-Square.